

# UNITARY REPRESENTATIONS OF NILPOTENT SUPER LIE GROUPS

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**ABSTRACT.** We show that irreducible unitary representations of nilpotent super Lie groups can be obtained by induction from a distinguished class of sub super Lie groups. These sub super Lie groups are natural analogues of polarizing subgroups that appear in classical Kirillov theory. We obtain a concrete geometric parametrization of irreducible unitary representations by nonnegative definite coadjoint orbits. As an application, we prove an analytic generalization of the Stone-von Neumann theorem for Heisenberg-Clifford super Lie groups.

## 1. INTRODUCTION

**1.1. Background.** One of the most elegant results in the theory of unitary representations is the Stone-von Neumann theorem, which yields a classification of irreducible unitary representations of the Heisenberg group. It is the starting point in the study of unitary representations of nilpotent Lie groups, in which it plays an essential role as well.

Kirillov's seminal work on unitary representations of nilpotent Lie groups showed that unitary representations can be obtained in a simple fashion, namely as induced representations from one-dimensional representations of certain subgroups which are called *polarizing subgroups*. From this, Kirillov deduced a well-behaved correspondence between irreducible unitary representations and coadjoint orbits.

Physicists are interested in unitary representations of Lie superalgebras and super Lie groups <sup>1</sup> and their applications, e.g. in the classification of free relativistic super particles in SUSY quantum mechanics (see [SaSt] and [FSZ]). Extensions of the Stone-von Neumann theorem to the Heisenberg-Clifford super Lie group, and the oscillator representation to the orthosymplectic case, have been studied widely by physicists as well as mathematicians (see [Ni], [Lo], and [BaGu]).

Nevertheless, much of the work done on infinite-dimensional unitary representations of super Lie groups treats representations algebraically, without addressing the analytic aspects. When the even part of the Lie superalgebra is a reductive Lie algebra (e.g., for classical simple Lie superalgebras)

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<sup>1</sup> We follow [DeMo] and [CCTV] in using the terms *super Lie group* and *sub super Lie group*. Nevertheless, instead of Deligne and Morgan's *super Lie algebra* we use the term *Lie superalgebra* merely because the latter is used in the literature more frequently.

the space of the representation can be identified with the space of  $K$ -finite analytic vectors of a unitary representation of the even part on a Hilbert space. This approach has been pursued in [FuNi]. However, this method is not applicable to more general super Lie groups, e.g., the nilpotent ones.

Motivated by establishing a rigorous formalism for Mackey-Wigner's little group method in the super setting, the authors of [CCTV] establish analytic foundations of the theory of unitary representations of super Lie groups. The key observation is that for infinite-dimensional representations, the action of the odd part of the Lie superalgebra is by *unbounded* operators, and thus one should consider densely defined operators. As shown in [CCTV], it turns out that the correct space to realize the action of the odd part is the (dense) subspace of *smooth* vectors (in the sense of [Kn, p. 52]) for the even part.

**1.2. Our main results.** The main goal of this work is to show that irreducible unitary representations (in the sense of [CCTV]) of nilpotent super Lie groups can be described in a way which is very similar to the classical work of Kirillov. More specifically, our results are as follows.

- (a) We generalize the notion of polarizing subalgebras of nilpotent Lie algebras to what we call *polarizing systems* in nilpotent super Lie groups. Let  $(N_0, \mathfrak{n})$  be a nilpotent super Lie group. A polarizing system of  $(N_0, \mathfrak{n})$  is a 6-tuple

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$$

where  $(M_0, \mathfrak{m})$  is a sub super Lie group of  $(N_0, \mathfrak{n})$ ,  $\Phi : (M_0, \mathfrak{m}) \rightarrow (C_0, \mathfrak{c})$  is a homomorphism onto a super Lie group of Clifford type, and  $\lambda \in \mathfrak{n}_0^*$ . (There are extra conditions which are stated in Definition 6.1.) We show that every irreducible unitary representation of a nilpotent super Lie group is induced from a special Clifford module associated to a polarizing system (see part (a) of Theorem 6.2). The latter module is said to be *consistent* with the polarizing system. Conversely, we prove that induction from a consistent representation of a polarizing system always results in an irreducible unitary representation (see Theorem 6.4). In other words, we show that induction yields the following surjective map:

$$\left\{ \begin{array}{l} \text{Ordered pairs of polarizing} \\ \text{systems of } (N_0, \mathfrak{n}) \text{ and their} \\ \text{consistent representations} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Irreducible unitary} \\ \text{representations of } (N_0, \mathfrak{n}) \\ \text{up to unitary equivalence} \end{array} \right\}$$

- (b) Given a  $\lambda \in \mathfrak{n}_0^*$ , we obtain a simple necessary and sufficient condition for the existence of a polarizing system  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  with a consistent representation. For every  $\lambda \in \mathfrak{n}_0^*$ , consider the symmetric bilinear form

$$B_\lambda : \mathfrak{n}_1 \times \mathfrak{n}_1 \rightarrow \mathbb{R}$$

defined by  $B_\lambda(X, Y) = \lambda([X, Y])$ . In Section 6.4 we prove that such a polarizing system with a consistent representation exists if and only if  $B_\lambda$  is nonnegative definite.

- (c) We obtain a concrete geometric parametrization of irreducible unitary representations of  $(N_0, \mathfrak{n})$ . Let

$$\mathfrak{n}_0^+ = \{ \lambda \in \mathfrak{n}_0^* \mid B_\lambda \text{ is nonnegative definite} \}.$$

It is easily checked that  $\mathfrak{n}_0^+$  is a union of coadjoint orbits. In Theorem 6.16 we prove that the inducing construction outlined above yields the following bijective correspondence:

$$\{ N_0\text{-orbits in } \mathfrak{n}_0^+ \} \longleftrightarrow \left\{ \begin{array}{l} \text{Irreducible unitary representations of} \\ (N_0, \mathfrak{n}) \text{ up to unitary equivalence} \\ \text{and parity change} \end{array} \right\}$$

- (d) As a simple application of our results, we obtain a proof of an analytic formulation of the Stone-von Neumann theorem for Heisenberg-Clifford super Lie groups (see Section 5.2). We believe that this analytic formulation is new. We would like to mention that in [Ro], the author studies a generalization of the Stone-von Neumann theorem to the super case. Our approach has the advantage that it yields a concrete statement based on a rigorous and more general notion of unitary representation for super Lie groups, and avoids the assumption that the odd part have even dimension.
- (e) A consequence of part (b) of Theorem 6.2 is a numerical invariant of irreducible unitary representations of nilpotent super Lie groups. The value of the invariant is a positive integer, and is equal to one if and only if the representation is *purely even*, i.e., in its  $\mathbb{Z}_2$ -grading the odd summand is trivial.

In conclusion, this work is yet another justification for fruitfulness of the approach pursued in [CCTV] to define and study unitary representations of super Lie groups rigorously.

**1.3. Organization of the paper.** This paper is organized as follows. Section 2 is devoted to recalling some basic definitions and facts about super Lie groups and their unitary representations. In Section 3 we recall the notion of induction of unitary representations from *special* sub super Lie groups which was introduced in [CCTV], and prove that it can be done in stages (see Proposition 3.1). Section 4 is devoted to studying the structure of nilpotent super Lie groups, proving a version of Kirillov's Lemma, and classification of representations of super Lie groups of Clifford type. Section 5.1 contains a technical but important result. In this section we prove that under certain conditions a unitary representation is induced from a sub super Lie group of codimension one. Although this result is analogous to one of Kirillov's original results, there are several delicate issues involving unbounded operators which need to be dealt with. Using the main result of Section 5.1, in Section 5.2 we obtain a proof of an analytic formulation of

the Stone-von Neumann theorem for Heisenberg-Clifford super Lie groups. In Section 6.1 we define polarizing systems, prove the surjectivity of the map from induced representations to irreducible representations, and show that if two polarizing systems yield the same representation then they correspond to the same coadjoint orbit. In Section 6.3 we prove the existence of a special kind of polarizing Lie subalgebra of  $\mathfrak{n}_0$ . This section is fairly technical and contains several lemmas, but the main point is to prove Lemma 6.10. In Section 6.4 we state and prove our main result on parametrization of representations by coadjoint orbits (see Theorem 6.16).

**1.4. Acknowledgement.** After the first draft of this article was written, we realized that M. Duflo had previously worked on the same problem and obtained similar results which were not published. We would like to thank M. Duflo for extremely illuminating conversations, and his encouragement to write this article.

## 2. PRELIMINARIES

**2.1. Notation and basic definitions.** Recall that a densely defined operator  $T$  on a Hilbert space  $\mathcal{H}$  is called symmetric if for every  $v, w \in D(T)$  we have  $\langle Tv, w \rangle = \langle v, Tw \rangle$ , where  $D(T)$  denotes the domain of  $T$ .

By a  $\mathbb{Z}_2$ -graded Hilbert space we mean a Hilbert space  $\mathcal{H}$  with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1.$$

A densely defined linear operator  $T$  on  $\mathcal{H}$  is called even (respectively, odd) if its domain  $D(T)$  is  $\mathbb{Z}_2$ -graded, i.e.,

$$D(T) = D(T)_0 \oplus D(T)_1$$

where for every  $i \in \{0, 1\}$  we have  $D(T)_i = D(T) \cap \mathcal{H}_i$ , and for every  $v \in D(T)_i$  we have  $Tv \in \mathcal{H}_i$  (respectively,  $Tv \in \mathcal{H}_{1-i}$ ).

For basic definitions and facts about Lie superalgebras, we refer the reader to [DeMo] and [Ka]. Unless explicitly stated otherwise, in this paper all Lie algebras and Lie superalgebras are over  $\mathbb{R}$ .

If  $\mathfrak{g}$  is a Lie superalgebra, its centre and universal enveloping algebra are denoted by  $\mathcal{Z}(\mathfrak{g})$  and  $\mathcal{U}(\mathfrak{g})$ . Similarly, the centre of a Lie group  $G$  is denoted by  $\mathcal{Z}(G)$ . If a Lie group  $G$  acts on a vector space  $\mathcal{V}$ , then the action of an element  $g \in G$  on a vector  $v \in \mathcal{V}$  is denoted by  $g \cdot v$ .

Following [DeMo], our definition of a super Lie group is based on the notion of a *Harish-Chandra pair*. One can define a super Lie group concretely as follows.

**Definition 2.1.** *A super Lie group is a pair  $(G_0, \mathfrak{g})$  with the following properties.*

- (a)  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  is a Lie superalgebra over  $\mathbb{R}$ .
- (b)  $G_0$  is a connected real Lie group with Lie algebra  $\mathfrak{g}_0$  which acts on  $\mathfrak{g}$  smoothly via  $\mathbb{R}$ -linear automorphisms.

- (c) *The action of  $G_0$  on  $\mathfrak{g}_0$  is the adjoint action. The adjoint action of  $\mathfrak{g}_0$  on  $\mathfrak{g}$  is the differential of the action of  $G_0$  on  $\mathfrak{g}$ .*

A super Lie group  $(H_0, \mathfrak{h})$  is called a *sub super Lie group* of a super Lie group  $(G_0, \mathfrak{g})$  if  $H_0$  is a Lie subgroup of  $G_0$  and  $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$  is a subalgebra<sup>2</sup> of  $\mathfrak{g}$  such that  $\mathfrak{h}_0$  is the Lie subalgebra of  $\mathfrak{g}_0$  corresponding to  $H_0$  and the action of  $H_0$  on  $\mathfrak{h}$  is the restriction of the action of  $G_0$  on  $\mathfrak{g}$ .

Let  $(G_0, \mathfrak{g})$  and  $(G'_0, \mathfrak{g}')$  be arbitrary super Lie groups. A homomorphism

$$\Phi : (G_0, \mathfrak{g}) \rightarrow (G'_0, \mathfrak{g}')$$

consists of a Lie group homomorphism from  $G_0$  to  $G'_0$  and a homomorphism of Lie superalgebras from  $\mathfrak{g}$  to  $\mathfrak{g}'$  which are compatible with each other. We say  $\Phi$  is surjective if both of these homomorphisms are surjective in the usual sense.

If  $(\pi, \mathcal{H})$  is a unitary representation of a Lie group  $G$  on a Hilbert space  $\mathcal{H}$ , then the subspace of smooth vectors of  $\mathcal{H}$  for the action of  $G$  is denoted by  $\mathcal{H}^\infty$ . The infinitesimal action of the Lie algebra of  $G$  on  $\mathcal{H}^\infty$  is denoted by  $\pi^\infty$ .

The definition of unitary representations of super Lie groups, which is given below, is originally introduced in [CCTV].

**Definition 2.2.** *A unitary representation of  $(G_0, \mathfrak{g})$  is a triple  $(\pi, \rho^\pi, \mathcal{H})$  such that  $\mathcal{H}$  is a  $\mathbb{Z}_2$ -graded Hilbert space endowed with a unitary representation  $\pi$  of  $G_0$ , and  $\rho^\pi : \mathfrak{g}_1 \rightarrow \text{End}_{\mathbb{C}}(\mathcal{H}^\infty)$  is an  $\mathbb{R}$ -linear map with the following properties.*

- (a) *For every  $g \in G_0$ ,  $\pi(g)$  is an even operator on  $\mathcal{H}$ .*
- (b) *For every  $X \in \mathfrak{g}_1$ ,  $\rho^\pi(X)$  is an odd linear operator. Moreover,  $\rho^\pi(X)$  is symmetric, i.e., for every  $v, w \in \mathcal{H}^\infty$ , we have*

$$\langle \rho^\pi(X)v, w \rangle = \langle v, \rho^\pi(X)w \rangle.$$

- (c) *For every  $X, Y \in \mathfrak{g}_1$  and  $v \in \mathcal{H}^\infty$ , we have*

$$\rho^\pi(X)\rho^\pi(Y)v + \rho^\pi(Y)\rho^\pi(X)v = -\sqrt{-1}\pi^\infty([X, Y])v.$$

- (d) *For every  $g \in G_0$  and  $X \in \mathfrak{g}_1$ , we have*

$$\rho^\pi(g \cdot X) = \pi(g)\rho^\pi(X)\pi(g^{-1}).$$

**Remark.** 1. One can combine  $\rho^\pi$  and  $\pi^\infty$  to obtain a representation of  $\mathfrak{g}$  in  $\mathcal{H}^\infty$  where an element  $X_0 + X_1 \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$  acts by  $\pi^\infty(X_0) + e^{\frac{\pi}{4}\sqrt{-1}}\rho^\pi(X_1)$ . Consequently, from [CCTV, Proposition 1] it follows that for every  $X \in \mathfrak{g}_0$ ,  $Y \in \mathfrak{g}_1$ , and  $v \in \mathcal{H}^\infty$  we have

$$\rho^\pi([X, Y])v = \pi^\infty(X)\rho^\pi(Y)v - \rho^\pi(Y)\pi^\infty(X)v.$$

2. From the closed graph theorem for Fréchet spaces, it follows that for every  $X \in \mathfrak{g}_1$ ,  $\rho^\pi(X)$  is a continuous operator on  $\mathcal{H}^\infty$ .

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<sup>2</sup> In this paper, instead of the term *sub super Lie algebra* of [DeMo] we use the abbreviation *subalgebra*.

Given two unitary representations  $(\pi, \rho^\pi, \mathcal{H})$  and  $(\pi', \rho^{\pi'}, \mathcal{H}')$  of  $(G_0, \mathfrak{g})$ , by an intertwining operator from  $(\pi, \rho^\pi, \mathcal{H})$  to  $(\pi', \rho^{\pi'}, \mathcal{H}')$  we mean an even bounded linear transformation  $T : \mathcal{H} \rightarrow \mathcal{H}'$  such that for every  $g \in G_0$  and  $X \in \mathfrak{g}_1$  we have  $T\pi(g) = \pi'(g)T$  and  $T\rho^\pi(X) = \rho^{\pi'}(X)T$ . (Note that if  $T\pi(g) = \pi'(g)T$  for every  $g \in G_0$ , then  $T\mathcal{H}^\infty \subseteq \mathcal{H}'^\infty$ .)

Two unitary representations  $(\pi, \rho^\pi, \mathcal{H})$  and  $(\pi', \rho^{\pi'}, \mathcal{H}')$  of  $(G_0, \mathfrak{g})$  are said to be unitarily equivalent if there exists an isometry  $T : \mathcal{H} \rightarrow \mathcal{H}'$  which is also an intertwining operator. Note that it follows that  $T\mathcal{H}^\infty = \mathcal{H}'^\infty$ .

From now on, to indicate that two unitary representations  $(\pi, \rho^\pi, \mathcal{H})$  and  $(\pi', \rho^{\pi'}, \mathcal{H}')$  are unitarily equivalent, we write

$$(\pi, \rho^\pi, \mathcal{H}) \simeq (\pi', \rho^{\pi'}, \mathcal{H}').$$

A unitary representation  $(\pi, \rho^\pi, \mathcal{H})$  of  $(G_0, \mathfrak{g})$  is called irreducible if  $\mathcal{H}$  does not have any proper  $(G_0, \mathfrak{g})$ -invariant closed  $\mathbb{Z}_2$ -graded subspaces. By [CCTV, Lemma 5], a representation  $(\pi, \rho^\pi, \mathcal{H})$  is irreducible if and only if every intertwining operator from  $(\pi, \rho^\pi, \mathcal{H})$  to itself is scalar.

From every unitary representation  $(\pi, \rho^\pi, \mathcal{H})$  we can obtain a new unitary representation  $(\pi, \rho^\pi, \Pi\mathcal{H})$  where  $\Pi$  is the parity change operator. The operator  $\Pi$  can be considered as a special case of tensor product, namely tensoring  $(\pi, \rho^\pi, \mathcal{H})$  with a trivial  $(0|1)$ -dimensional representation. The unitary representations  $(\pi, \rho^\pi, \mathcal{H})$  and  $(\pi, \rho^\pi, \Pi\mathcal{H})$  are said to be the same up to parity change. Note that they are not necessarily unitarily equivalent.

From now on, to indicate that two unitary representations  $(\pi, \rho^\pi, \mathcal{H})$  and  $(\pi', \rho^{\pi'}, \mathcal{H}')$  are identical up to unitary equivalence and parity change, we write

$$(\pi, \rho^\pi, \mathcal{H}) \asymp (\pi', \rho^{\pi'}, \mathcal{H}').$$

**2.2. Stability of unitary representations.** A remarkable feature of unitary representations as defined in Definition 2.2 is their stability, i.e. that one can replace the space  $\mathcal{H}^\infty$  with a variety of dense and invariant subspaces. Stability is needed even for justifying that the restriction of a unitary representation of a super Lie group  $(G_0, \mathfrak{g})$  to a sub super Lie group  $(H_0, \mathfrak{h})$  is well defined, i.e., that the restriction determines a unique unitary representation up to unitary equivalence. The result that justifies the latter statement is [CCTV, Proposition 2]. For the reader's convenience, and for future reference in this article, we would like to record a slightly simplified formulation of the statement of this proposition.

**Proposition 2.3.** [CCTV, Proposition 2] *Let  $(G_0, \mathfrak{g})$  be a super Lie group and  $(\pi, \mathcal{H})$  be a unitary representation of  $G_0$ . Suppose  $\mathcal{B}$  is a dense,  $\mathbb{Z}_2$ -graded, and  $G_0$ -invariant subspace of  $\mathcal{H}$ , and  $\{\rho(X)\}_{X \in \mathfrak{g}_1}$  is a family of densely defined linear operators on  $\mathcal{H}$  with the following properties.*

- (a)  $\mathcal{B} \subseteq \mathcal{H}^\infty$ .
- (b) If  $X \in \mathfrak{g}_1$ , then  $\mathcal{B} \subseteq D(\rho(X))$ .
- (c)  $\rho(X)$  is symmetric for every  $X \in \mathfrak{g}_1$ .

- (d) For every  $X \in \mathfrak{g}_1$  and  $i \in \{0, 1\}$  we have  $\rho(X)\mathcal{B}_i \subseteq \mathcal{H}_{1-i}$ .
- (e) If  $X, Y \in \mathfrak{g}_1$  and  $a, b \in \mathbb{R}$  then  $\rho(aX + bY) = a\rho(X) + b\rho(Y)$ .
- (f)  $\pi(g)\rho(X)\pi(g^{-1}) = \rho(g \cdot X)$  for all  $g \in G_0$  and  $X \in \mathfrak{g}_1$ .
- (g) For every  $X, Y \in \mathfrak{g}_1$  we have  $\rho(X)\mathcal{B} \subseteq D(\rho(Y))$ .
- (h) For every  $X, Y \in \mathfrak{g}_1$  and  $v \in \mathcal{B}$  we have

$$\rho(X)\rho(Y)v + \rho(Y)\rho(X)v = -\sqrt{-1}\pi^\infty([X, Y])v.$$

Then the following statements hold.

- (i) For every  $X \in \mathfrak{g}_1$ , the operator  $\rho(X)$  is essentially self adjoint, and the closure  $\overline{\rho(X)}$  of  $\rho(X)$  satisfies  $\mathcal{H}^\infty \subseteq D(\overline{\rho(X)})$ .
- (ii) Suppose that for every  $X \in \mathfrak{g}_1$  and  $v \in \mathcal{H}^\infty$ , we set  $\rho^\pi(X)v = \overline{\rho(X)}v$ . Then for every  $X \in \mathfrak{g}_1$  we have  $\rho^\pi(X) \in \text{End}_{\mathbb{C}}(\mathcal{H}^\infty)$ . Moreover,  $(\pi, \rho^\pi, \mathcal{H})$  is a unitary representation of  $(G_0, \mathfrak{g})$ .
- (iii) Let  $(\pi', \rho^{\pi'}, \mathcal{H})$  be a unitary representation of  $(G_0, \mathfrak{g})$  in the same  $\mathbb{Z}_2$ -graded Hilbert space  $\mathcal{H}$ . Suppose that for every  $g \in G_0$  we have  $\pi'(g) = \pi(g)$ , and for every  $X \in \mathfrak{g}_1$  and  $v \in \mathcal{B}$  we have  $\rho^{\pi'}(X)v = \rho^\pi(X)v$ . Then  $(\pi', \rho^{\pi'}, \mathcal{H}) \simeq (\pi, \rho^\pi, \mathcal{H})$ , and the intertwining isometry  $T : \mathcal{H} \rightarrow \mathcal{H}$  yielding this unitary equivalence is the identity map.

We conclude this section with a simple but useful lemma about polarizing subalgebras. Let  $\mathfrak{g}$  be a Lie algebra and fix  $\lambda \in \mathfrak{g}^*$ . Recall that a Lie subalgebra  $\mathfrak{m}$  of  $\mathfrak{g}$  is called a polarizing subalgebra corresponding to  $\lambda$  if  $\mathfrak{m}$  is a maximal isotropic subspace of  $\mathfrak{g}$  for the skew-symmetric bilinear form  $\omega_\lambda : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  defined by  $\omega_\lambda(X, Y) = \lambda([X, Y])$ .

**Lemma 2.4.** *Let  $\mathfrak{g}$  be a nilpotent Lie algebra,  $\lambda \in \mathfrak{g}^*$ , and  $\mathfrak{m} \subseteq \mathfrak{g}$  be a Lie subalgebra. If  $\lambda \neq 0$  and  $\mathfrak{m}$  is a polarizing subalgebra of  $\mathfrak{g}$  corresponding to  $\lambda$ , then there exists an  $X \in \mathfrak{m}$  such that  $\lambda(X) \neq 0$ .*

*Proof.* Suppose, on the contrary, that  $\mathfrak{m} \subseteq \ker \lambda$ . Then from  $\lambda \neq 0$  it follows that  $\mathfrak{g} \not\supseteq \mathfrak{m}$ . If

$$N_{\mathfrak{g}}(\mathfrak{m}) = \{Y \in \mathfrak{g} \mid [Y, \mathfrak{m}] \subseteq \mathfrak{m}\}$$

then  $N_{\mathfrak{g}}(\mathfrak{m})$  is a Lie subalgebra of  $\mathfrak{g}$  and  $N_{\mathfrak{g}}(\mathfrak{m}) \supsetneq \mathfrak{m}$ . Choose an  $X \in N_{\mathfrak{g}}(\mathfrak{m})$  such that  $X \notin \mathfrak{m}$  and set  $\mathfrak{m}' = \mathfrak{m} \oplus \mathbb{R}X$ . It is easy to check that  $[\mathfrak{m}', \mathfrak{m}'] \subseteq \mathfrak{m}$  and thus  $\lambda([\mathfrak{m}', \mathfrak{m}']) = \{0\}$  which contradicts maximality of dimension of  $\mathfrak{m}$ .  $\square$

### 3. SPECIAL INDUCTION

**3.1. Realization of the induced representation.** Let  $(G_0, \mathfrak{g})$  be a super Lie group and  $(H_0, \mathfrak{h})$  be a sub super Lie group of  $(G_0, \mathfrak{g}_0)$ , i.e.,  $H_0 \subseteq G_0$  and  $\mathfrak{h} \subseteq \mathfrak{g}$ . As in [CCTV, §3.2], we assume that  $(H_0, \mathfrak{h})$  is *special*, i.e., that  $\mathfrak{h}_1 = \mathfrak{g}_1$ . For every unitary representation  $(\sigma, \rho^\sigma, \mathcal{K})$  of  $(H_0, \mathfrak{h})$ , the representation of  $(G_0, \mathfrak{g})$  induced from  $(\sigma, \rho^\sigma, \mathcal{K})$  is defined in [CCTV, §3]. We recall the definition of special induction only in a case which we need in this paper, i.e., when the Lie groups  $G_0$  and  $H_0$  are unimodular. In

this case, to define the representation  $(\pi, \rho^\pi, \mathcal{H})$  of  $(G_0, \mathfrak{g})$  induced from  $(\sigma, \rho^\sigma, \mathcal{K})$ , one fixes a  $G_0$ -invariant measure  $\mu$  on  $H_0 \backslash G_0$  and defines  $\mathcal{H}$  as the space of measurable functions  $f : G_0 \rightarrow \mathcal{K}$  such that

- (a) For any  $g \in G_0$  and  $h \in H_0$ , we have  $f(hg) = \sigma(h)f(g)$ .
- (b)  $\int_{H_0 \backslash G_0} \|f(g)\|^2 d\mu < \infty$ .

The action of every  $g \in G_0$  on every  $f \in \mathcal{H}$  is the usual right regular representation, i.e.,

$$\text{if } g, g' \in G_0 \text{ then } (\pi(g)f)(g') = f(g'g).$$

Recall that  $\mathcal{H}^\infty$  is the space of smooth vectors of  $(\pi, \mathcal{H})$ . It is well-known that

$$\mathcal{H}^\infty \subseteq C^\infty(G_0, \mathcal{K}),$$

where  $C^\infty(G_0, \mathcal{K})$  denotes the space of smooth functions  $f : G_0 \rightarrow \mathcal{K}$  (see [Po, Theorem 5.1] or [CG, Theorem A.1.4]). Moreover, one can check that for every  $f \in \mathcal{H}^\infty$ , we have  $f(G_0) \subseteq \mathcal{K}^\infty$ .

Let  $\mathcal{H}^{\infty, c}$  be the space consisting of functions  $f : G_0 \rightarrow \mathcal{K}$  such that

$$f \in \mathcal{H} \cap C^\infty(G_0, \mathcal{K})$$

and  $\text{Supp}(f)$  is compact modulo  $H_0$ . It is shown in [CCTV, Proposition 4] that  $\mathcal{H}^{\infty, c} \subseteq \mathcal{H}^\infty$ , the subspace  $\mathcal{H}^{\infty, c}$  is dense in  $\mathcal{H}$ , and for every  $X \in \mathfrak{g}_0$  we have  $\pi^\infty(X)\mathcal{H}^{\infty, c} \subseteq \mathcal{H}^{\infty, c}$ . The action of  $\mathfrak{g}_1$  is initially defined on  $\mathcal{H}^{\infty, c}$ . For every  $X \in \mathfrak{g}_1$  and  $f \in \mathcal{H}^{\infty, c}$ , one defines

$$(3.1) \quad (\rho^\pi(X)f)(g) = \rho^\sigma(g \cdot X)(f(g)).$$

From Proposition 2.3 it follows that the domain of the closure of  $\rho^\pi(X)$  contains  $\mathcal{H}^\infty$ , and consequently the induced representation  $(\pi, \rho^\pi, \mathcal{H})$  is well-defined. We will denote the latter representation by

$$\text{Ind}_{(H_0, \mathfrak{h})}^{(G_0, \mathfrak{g})}(\sigma, \rho^\sigma, \mathcal{K}).$$

**3.2. Special induction in stages.** A basic but important property of special induction is that it can be done in stages. The proof of this property is not difficult, but it is not mentioned in [CCTV] explicitly. For the reader's convenience, we would like to sketch it below.

**Proposition 3.1.** *Suppose that  $(G_0, \mathfrak{g})$  is a super Lie group,  $(K_0, \mathfrak{k})$  is a special sub super Lie group of  $(G_0, \mathfrak{g})$ , and  $(H_0, \mathfrak{h})$  is a special sub super Lie group of  $(H_0, \mathfrak{h})$ . Assume that  $G_0, K_0$ , and  $H_0$  are unimodular, and let  $(\sigma, \rho^\sigma, \mathcal{K})$  be a unitary representation of  $(H_0, \mathfrak{h})$ . Then*

$$(3.2) \quad \text{Ind}_{(H_0, \mathfrak{h})}^{(G_0, \mathfrak{g})}(\sigma, \rho^\sigma, \mathcal{K}) \simeq \text{Ind}_{(K_0, \mathfrak{k})}^{(G_0, \mathfrak{g})} \text{Ind}_{(H_0, \mathfrak{h})}^{(K_0, \mathfrak{k})}(\sigma, \rho^\sigma, \mathcal{K}).$$

*Proof.* Set

$$(\pi, \rho^\pi, \mathcal{H}) = \text{Ind}_{(H_0, \mathfrak{h})}^{(G_0, \mathfrak{g})}(\sigma, \rho^\sigma, \mathcal{K}), \quad (\eta, \rho^\eta, \mathcal{L}) = \text{Ind}_{(H_0, \mathfrak{h})}^{(K_0, \mathfrak{k})}(\sigma, \rho^\sigma, \mathcal{K}),$$

and

$$(\nu, \rho^\nu, \mathcal{V}) = \text{Ind}_{(K_0, \mathfrak{k})}^{(G_0, \mathfrak{g})}(\eta, \rho^\eta, \mathcal{L}).$$

Thus  $\mathcal{H}$ ,  $\mathcal{L}$ , and  $\mathcal{V}$  are function spaces introduced in Section 3.1 which realize the corresponding induced representations. Let  $\mathcal{H}^{\infty,c}$  be the subspace of  $\mathcal{H}$  defined in Section 3.1. We define  $\mathcal{L}^{\infty,c}$  and  $\mathcal{V}^{\infty,c}$  similarly. The intertwining map

$$T : (\pi, \mathcal{H}) \rightarrow (\nu, \mathcal{V})$$

is given in [Ma, §4]. We recall the definition of  $T$ . Given a function  $f : G_0 \rightarrow \mathcal{K}$  such that  $f \in \mathcal{H}^{\infty,c}$ , the function  $Tf : G_0 \rightarrow \mathcal{L}$  is obtained as follows. For every  $g \in G_0$  and  $k \in K_0$ ,

$$(Tf(g))(k) = f(kg).$$

One can normalize the involved measures such that for every  $f \in \mathcal{H}^{\infty,c}$ , we have  $\|Tf\| = \|f\|$ .

Fix an  $f \in \mathcal{H}^{\infty,c}$  and a  $V \in \mathfrak{g}_1$ . Since  $f$  is a smooth vector for  $\pi = \text{Ind}_{H_0}^{G_0} \sigma$  and  $T$  is an intertwining isometry,  $Tf$  is a smooth vector for

$$\nu = \text{Ind}_{H_0}^{G_0} \text{Ind}_{K_0}^{H_0} \sigma.$$

By Proposition 2.3, to prove Proposition 3.1 it suffices to show that

$$(3.3) \quad \text{for every } f \in \mathcal{H}^{\infty,c} \text{ and } V \in \mathfrak{g}_1, \quad T\rho^\pi(V)f = \rho^\nu(V)Tf.$$

Since  $\text{Supp}(f)$  is compact modulo  $H_0$ , it follows readily that  $\text{Supp}(Tf)$  is compact modulo  $K_0$ , and for every  $g \in G_0$  the support of

$$Tf(g) : K_0 \rightarrow \mathcal{K}$$

is compact modulo  $H_0$ . From [CCTV, Proposition 4] it follows that  $Tf \in \mathcal{V}^{\infty,c}$  and for every  $g \in G_0$  we have  $Tf(g) \in \mathcal{L}^{\infty,c}$ . By the definition of special induction given in Section 3.1, the action of  $\rho^\nu(V)$  on  $Tf$  is given by (3.1). For the same reason, the action of  $\rho^\eta(V)$  on  $Tf(g)$  is given by (3.1). Thus for every  $g \in G_0$ , and  $k \in K_0$ ,

$$\begin{aligned} \left( (\rho^\nu(V)Tf)(g) \right)(k) &= \left( \rho^\eta(g \cdot V)(Tf(g)) \right)(k) \\ &= \rho^\sigma(kg \cdot V) \left( (Tf(g))(k) \right) = \rho^\sigma(kg \cdot V)(f(kg)). \end{aligned}$$

To finish the proof of (3.3) note that

$$\begin{aligned} \left( (T\rho^\pi(V)f)(g) \right)(k) &= (\rho^\pi(V)f)(kg) \\ &= \rho^\sigma(kg \cdot V)(f(kg)) = \left( (\rho^\nu(V)Tf)(g) \right)(k). \end{aligned}$$

□

#### 4. REDUCED FORMS AND SUPER LIE GROUPS OF CLIFFORD TYPE

**4.1. The reduced form of a super Lie group.** Contrary to the case of locally compact groups, super Lie groups do not necessarily have faithful representations. The next lemma presents a simple but key example of such elements.

**Lemma 4.1.** *Let  $(G_0, \mathfrak{g})$  be a super Lie group and  $(\pi, \rho^\pi, \mathcal{H})$  be a unitary representation of  $(G_0, \mathfrak{g})$ . If  $X_1, \dots, X_m \in \mathfrak{g}_1$  satisfy*

$$\sum_{i=1}^m [X_i, X_i] = 0$$

*then  $\rho^\pi(X_i) = 0$  for every  $1 \leq i \leq m$ .*

*Proof.* We have

$$\sum_{i=1}^m \rho^\pi(X_i)^2 = -\frac{\sqrt{-1}}{2} \sum_{i=1}^m \pi^\infty([X_i, X_i]) = -\frac{\sqrt{-1}}{2} \pi^\infty\left(\sum_{i=1}^m [X_i, X_i]\right) = 0.$$

Since every  $\rho^\pi(X_i)$  is symmetric, for every  $v \in \mathcal{H}^\infty$  we have

$$\sum_{i=1}^m \langle \rho^\pi(X_i)v, \rho^\pi(X_i)v \rangle = \langle v, \sum_{i=1}^m \rho^\pi(X_i)^2 v \rangle = 0.$$

Therefore for every  $1 \leq i \leq m$  we have  $\langle \rho^\pi(X_i)v, \rho^\pi(X_i)v \rangle = 0$ , which implies that  $\rho^\pi(X_i)v = 0$ . □

The proof of the following proposition is easy by induction.

**Proposition 4.2.** *Let  $(G_0, \mathfrak{g})$  be a super Lie group. Let  $\mathfrak{a}^{(1)}$  be the ideal of  $\mathfrak{g}$  generated by all  $X \in \mathfrak{g}_1$  such that  $[X, X] = 0$ . For every  $m > 1$ , let  $\mathfrak{a}^{(m)}$  be the ideal of  $\mathfrak{g}$  generated by elements  $X \in \mathfrak{g}_1$  such that  $[X, X] \in \mathfrak{a}^{(m-1)}$ . Then for every unitary representation  $(\pi, \rho^\pi, \mathcal{H})$  of  $(G_0, \mathfrak{g})$ , the  $\mathbb{Z}_2$ -graded ideal  $\bigcup_{m=1}^\infty \mathfrak{a}^{(m)}$  acts trivially on  $\mathcal{H}$ , i.e.,  $\rho^\pi(X) = 0$  if  $X \in \mathfrak{g}_1 \cap \bigcup_{m=1}^\infty \mathfrak{a}^{(m)}$ , and  $\pi^\infty(X) = 0$  if  $X \in \mathfrak{g}_0 \cap \bigcup_{m=1}^\infty \mathfrak{a}^{(m)}$ .*

Note that  $\mathfrak{a}^{(1)} \subseteq \mathfrak{a}^{(2)} \subseteq \dots$  and the set  $\mathfrak{a}[\mathfrak{g}] = \bigcup_{m=1}^\infty \mathfrak{a}^{(m)}$  is a  $\mathbb{Z}_2$ -graded ideal of  $\mathfrak{g}$ . Therefore we have  $\mathfrak{a}[\mathfrak{g}] = \mathfrak{a}[\mathfrak{g}]_0 \oplus \mathfrak{a}[\mathfrak{g}]_1$ . Let  $A_0$  be the normal subgroup of  $G_0$  with Lie algebra  $\mathfrak{a}[\mathfrak{g}]_0$ . Then  $(A_0, \mathfrak{a}[\mathfrak{g}])$  is a sub super Lie group of  $(G_0, \mathfrak{g})$ . Moreover, setting  $\overline{G}_0 = G_0/A_0$  and  $\overline{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}[\mathfrak{g}]$  we obtain a nilpotent super Lie group  $(\overline{G}_0, \overline{\mathfrak{g}})$ .

From now on, the super Lie group  $(\overline{G}_0, \overline{\mathfrak{g}})$  will be called the *reduced form* of  $(G_0, \mathfrak{g})$ . Obviously, the categories of unitary representations of  $(G_0, \mathfrak{g})$  and  $(\overline{G}_0, \overline{\mathfrak{g}})$  are equivalent.

If  $(G_0, \mathfrak{g})$  is a super Lie group with the property that  $\mathfrak{a}[\mathfrak{g}] = \{0\}$ , then the super Lie group  $(G_0, \mathfrak{g})$  and the Lie superalgebra  $\mathfrak{g}$  are called *reduced*.

**4.2. Heisenberg-Clifford super Lie groups.** In this section we introduce an important example of nilpotent super Lie groups which will be used in the rest of this paper.

Let  $(\mathfrak{w}, \omega)$  be a super symplectic vector space, i.e., a  $\mathbb{Z}_2$ -graded vector space  $\mathfrak{w} = \mathfrak{w}_0 \oplus \mathfrak{w}_1$  endowed with a nondegenerate bilinear form

$$\omega : \mathfrak{w} \times \mathfrak{w} \rightarrow \mathbb{R}$$

with the following properties.

- (a)  $\omega(\mathfrak{w}_0, \mathfrak{w}_1) = \omega(\mathfrak{w}_1, \mathfrak{w}_0) = \{0\}$ .
- (b) Restriction of  $\omega$  to  $\mathfrak{w}_0$  is a symplectic form.
- (c) Restriction of  $\omega$  to  $\mathfrak{w}_1$  is a symmetric form.

Consider the  $\mathbb{Z}_2$ -graded vector space

$$\mathfrak{n} = \mathfrak{w} \oplus \mathbb{R}$$

where  $\mathfrak{n}_0 = \mathfrak{w}_0 \oplus \mathbb{R}$  and  $\mathfrak{n}_1 = \mathfrak{w}_1$ . We define a (super)bracket on  $\mathfrak{n}$  as follows. For every  $P, Q \in \mathfrak{w}$  and  $a, b \in \mathbb{R}$ , we set

$$[(P, a), (Q, b)] = (0, \omega(P, Q)).$$

One can easily check that with this bracket  $\mathfrak{n}$  becomes a Lie superalgebra. The latter Lie superalgebra is called a Heisenberg-Clifford Lie superalgebra. If  $N_0$  denotes the simply connected nilpotent Lie group with Lie algebra  $\mathfrak{n}_0$ , then the super Lie group  $(N_0, \mathfrak{n})$  is called a Heisenberg-Clifford super Lie group.

It may sometimes be more convenient to work with an explicit basis of the Heisenberg-Clifford Lie superalgebra. One can always find a basis

$$(4.1) \quad \{Z, X_1, \dots, X_m, Y_1, \dots, Y_m, V_1, \dots, V_n\}$$

of  $\mathfrak{n}$  such that

- (a)  $\mathfrak{n}_0 = \text{Span}_{\mathbb{R}}\{Z, X_1, \dots, X_m, Y_1, \dots, Y_m\}$  and  $\mathfrak{n}_1 = \text{Span}_{\mathbb{R}}\{V_1, \dots, V_n\}$ .
- (b) For every  $1 \leq i \leq m$  we have  $[X_i, Y_i] = Z$ .
- (c) For every  $1 \leq j \leq n$  we have

$$(4.2) \quad [V_j, V_j] = c_j Z$$

where  $c_j \in \{1, -1\}$ .

- (d)  $Z \in \mathcal{Z}(\mathfrak{n})$ .

**4.3. Nilpotent supergroups.** Recall that a Lie superalgebra  $\mathfrak{g}$  is called nilpotent if  $\mathfrak{g}$  appears in its own upper central series. (Equivalently,  $\mathfrak{g}$  is called nilpotent if its lower central series has only finitely many nonzero terms.)

In this paper, a super Lie group  $(N_0, \mathfrak{n})$  is called nilpotent if it has the following properties.

- (a)  $\mathfrak{n}$  is a nilpotent Lie superalgebra.
- (b)  $N_0$  is a connected, simply connected, nilpotent Lie group.

It follows that the exponential map  $\exp : \mathfrak{n}_0 \rightarrow N_0$  is an analytic diffeomorphism which results in a bijective correspondence between Lie subgroups and Lie subalgebras.

**4.4. Structure of reduced forms.** Our next task is to state and prove a generalization of Kirillov's lemma [CG, Lemma 1.1.12]. The proof of this generalization is a slight modification of that of the original result. Recall that  $\mathcal{Z}(\mathfrak{n})$  denotes the centre of  $\mathfrak{n}$ .

**Definition 4.3.** A nilpotent Lie superalgebra  $\mathfrak{n}$  is said to be of Clifford type if one of the following properties hold.

- (a)  $\mathfrak{n} = \{0\}$ .
- (b)  $\mathfrak{n}$  is a Heisenberg-Clifford Lie superalgebra such that  $\dim \mathfrak{n}_0 = 1$  and the restriction of  $\omega$  to  $\mathfrak{n}_1$  is positive definite.

In other words,  $\mathfrak{n}$  is of Clifford type if either  $\mathfrak{n} = \{0\}$  or  $\mathfrak{n}$  satisfies both of the following properties.

- (a)  $\dim \mathfrak{n}_0 = 1$  and  $\mathcal{Z}(\mathfrak{n}) = \mathfrak{n}_0$ .
- (b) There exists a basis

$$(4.3) \quad \{Z, V_1, \dots, V_l\}$$

of  $\mathfrak{n}$  such that  $Z \in \mathcal{Z}(\mathfrak{n}_0)$ ,  $V_1, \dots, V_l \in \mathfrak{n}_1$ , and for every  $1 \leq i \leq j \leq l$  we have  $[V_i, V_j] = \delta_{i,j}Z$ .

A nilpotent super Lie group  $(N_0, \mathfrak{n})$  is said to be of Clifford type whenever  $\mathfrak{n}$  is of Clifford type.

Note that the zero-dimensional Lie superalgebra and the (unique) Lie superalgebra  $\mathfrak{n}$  which satisfies  $\dim \mathfrak{n} = \dim \mathfrak{n}_0 = 1$  are also considered to be of Clifford type. Up to parity change, irreducible unitary representations of their corresponding super Lie groups are one-dimensional and purely even. Up to parity change, trivial representation is the only such representation of the first case. For the second case, these representations are unitary characters of the even part.

**Proposition 4.4.** *Let  $\mathfrak{n}$  be a reduced nilpotent Lie superalgebra which satisfies  $\dim \mathfrak{n} > 1$  and  $\dim \mathcal{Z}(\mathfrak{n}) = 1$ . Then exactly one of the following two statements is true.*

- (a) *There exist three nonzero elements  $X, Y, Z \in \mathfrak{n}_0$  such that*

$$\mathfrak{n} = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathfrak{w}$$

*where  $\mathfrak{w} = \mathfrak{w}_0 \oplus \mathfrak{w}_1$  is a  $\mathbb{Z}_2$ -graded subspace of  $\mathfrak{n}$ ,  $[X, Y] = Z$ , and  $Z \in \mathcal{Z}(\mathfrak{n})$ . Moreover, the vector space*

$$\mathfrak{n}' = \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathfrak{w}$$

*is a subalgebra of  $\mathfrak{n}$ , and  $Y \in \mathcal{Z}(\mathfrak{n}')$ .*

- (b)  *$\mathfrak{n}$  is a Lie superalgebra of Clifford type.*

*Proof.* Obviously  $\mathcal{Z}(\mathfrak{n}) \subseteq \mathfrak{n}_0$ , since if  $X \neq 0$  and  $X \in \mathfrak{n}_1 \cap \mathcal{Z}(\mathfrak{n})$ , then  $[X, X] = 0$  which contradicts the assumption that  $\mathfrak{n}$  is reduced.

Fix an arbitrary nonzero  $Z \in \mathcal{Z}(\mathfrak{n})$ . Since  $\mathfrak{n}$  is nilpotent, we have

$$\mathcal{Z}(\mathfrak{n}/\mathcal{Z}(\mathfrak{n})) \neq \{0\}.$$

Let  $\mathcal{Z}^1(\mathfrak{n})$  denote the  $\mathbb{Z}_2$ -graded ideal of  $\mathfrak{n}$  which corresponds to  $\mathcal{Z}(\mathfrak{n}/\mathcal{Z}(\mathfrak{n}))$  via the quotient map  $\mathfrak{q} : \mathfrak{n} \rightarrow \mathfrak{n}/\mathcal{Z}(\mathfrak{n})$ . Obviously  $\mathcal{Z}^1(\mathfrak{n}) \supsetneq \mathcal{Z}(\mathfrak{n})$ .

First assume that  $\mathcal{Z}^1(\mathfrak{n}) \cap \mathfrak{n}_0 \not\subseteq \mathcal{Z}(\mathfrak{n})$ . We show that statement (a) of the proposition holds. Choose an arbitrary  $Y \in \mathcal{Z}^1(\mathfrak{n}) \cap \mathfrak{n}_0$  such that  $Y \notin \mathcal{Z}(\mathfrak{n})$ , and consider the map  $\text{ad}_Y : \mathfrak{n} \rightarrow \mathcal{Z}(\mathfrak{n})$ . Since  $Y \in \mathfrak{n}_0$  and  $Y \notin \mathcal{Z}(\mathfrak{n})$ , there exists an element  $X \in \mathfrak{n}_0$  such that  $[X, Y] \neq 0$ . After an appropriate

rescaling, we can assume that  $[X, Y] = Z$ . We can now take  $\mathfrak{w}$  to be a  $\mathbb{Z}_2$ -graded complement of  $\mathbb{R}Y \oplus \mathbb{R}Z$  in  $\mathfrak{n}'$ , where

$$\mathfrak{n}' = \{ V \in \mathfrak{n} \mid [V, Y] = 0 \}.$$

Next assume that  $\mathcal{Z}^1(\mathfrak{n}) \cap \mathfrak{n}_0 \subseteq \mathcal{Z}(\mathfrak{n})$ . We use induction on  $\dim \mathfrak{n}$  to show that  $\mathfrak{n}$  is of Clifford type. Choose an arbitrary nonzero  $V_1 \in \mathcal{Z}^1(\mathfrak{n}) \cap \mathfrak{n}_1$ . Since  $\mathfrak{n}$  is reduced, after an appropriate rescaling we can assume that  $[V_1, V_1] = \pm Z$ . If  $\dim \mathfrak{n} = 2$ , then the proof is complete. Next assume  $\dim \mathfrak{n} > 2$ . Set

$$\mathfrak{n}' = \ker(\text{ad}_{V_1}).$$

Obviously  $\mathfrak{n}'$  is a subalgebra of  $\mathfrak{n}$  and  $\mathfrak{n} = \mathfrak{n}' \oplus \mathbb{R}V_1$  as vector spaces. It is easy to see that  $\mathfrak{n}'$  is reduced,  $\mathcal{Z}(\mathfrak{n}') = \mathcal{Z}(\mathfrak{n})$ , and  $\mathcal{Z}^1(\mathfrak{n}') = \mathcal{Z}^1(\mathfrak{n}) \cap \mathfrak{n}'$ . Therefore  $\dim \mathcal{Z}(\mathfrak{n}') = 1$  and  $\mathcal{Z}^1(\mathfrak{n}') \cap \mathfrak{n}'_0 \subseteq \mathcal{Z}(\mathfrak{n}')$ . By induction hypothesis, there exists a basis  $\{Z', V_2, \dots, V_l\}$  for  $\mathfrak{n}'$  such that  $Z' \in \mathcal{Z}(\mathfrak{n}')$ ,  $V_l \in \mathfrak{n}'_1$  for every  $l > 1$ , and  $[V_i, V_j] = \delta_{i,j}Z'$  for every  $1 < i \leq j \leq l$ . After rescaling  $V_1$  appropriately, we have  $[V_1, V_1] = \pm Z'$ . If  $[V_1, V_1] = Z'$ , then the proof is complete. If  $[V_1, V_1] = -Z'$ , then it follows that  $[V_1 + V_2, V_1 + V_2] = 0$ , contradicting the assumption that  $\mathfrak{n}$  is reduced.  $\square$

**4.5. Representations of super Lie groups of Clifford type.** Throughout this section, we assume that  $(C_0, \mathfrak{c})$  is a super Lie group of Clifford type such that  $\mathfrak{c} \neq \{0\}$ . Let  $\{Z, V_1, \dots, V_l\}$  be the basis of  $\mathfrak{c}$  given in (4.3), and  $(\pi, \rho^\pi, \mathcal{H})$  be an irreducible unitary representation of  $(C_0, \mathfrak{c})$ . By [CCTV, Lemma 5], the action of  $\rho^\pi(Z)$  is via multiplication by a scalar. It follows that  $\mathcal{H}^\infty = \mathcal{H}$ , i.e., for every  $1 \leq i \leq l$  we have

$$(4.4) \quad D(\rho^\pi(V_i)) = \mathcal{H}.$$

Fix  $1 \leq i \leq l$ . Since  $\rho^\pi(V_i)$  is symmetric, it is closable [Co, p. 316]. Thus (4.4) implies that  $\rho^\pi(V_i)$  is a closed operator. Consequently, by the closed graph theorem,  $\rho^\pi(V_i)$  is a bounded, self adjoint operator.

Since  $\rho^\pi(V_1)$  is a self adjoint operator and  $\pi^\infty(Z) = 2\sqrt{-1}\rho^\pi(V_1)^2$ , it follows that for every  $v \in \mathcal{H}$  we have

$$\pi^\infty(Z)v = a\sqrt{-1}v$$

where  $a \geq 0$ .

If  $a = 0$ , then for every  $V \in \mathfrak{c}_1$  the symmetric operator  $\rho^\pi(V)$  satisfies  $\rho^\pi(V)^2 = 0$ , and it follows immediately that  $\rho^\pi(V) = 0$ . Therefore  $(\pi, \rho^\pi, \mathcal{H})$  is a trivial representation.

Next suppose  $a > 0$ . Let  $\langle Z - a \rangle$  denote the two-sided ideal of  $\mathcal{U}(\mathfrak{c})$  generated by  $Z - a$ . We can set  $\rho(V) = \rho^\pi(V)$  for every  $V \in \mathfrak{c}_1$ , and then extend  $\rho$  to a homomorphism

$$\rho : \mathcal{U}(\mathfrak{c}) / \langle Z - a \rangle \rightarrow \text{End}_{\mathbb{C}}(\mathcal{H})$$

of associative (super)algebras. In this fashion, from a representation of  $(C_0, \mathfrak{c})$  we obtain a representation of  $\mathcal{U}(\mathfrak{c}) / \langle Z - a \rangle$  on a complex  $\mathbb{Z}_2$ -graded vector space.

Fix a nonzero vector  $v \in \mathcal{H}_0$  and consider the subspace  $\mathcal{W} \subseteq \mathcal{H}$  defined by

$$\mathcal{W} = \text{Span}_{\mathbb{C}}\{\rho(W)v \mid W \in \mathcal{U}(\mathfrak{c})\}.$$

Since  $\mathcal{U}(\mathfrak{c})$  is finite dimensional,  $\mathcal{W}$  is finite dimensional as well, and hence it is a closed subspace of  $\mathcal{H}$ . It is easily seen that  $\mathcal{W}$  is a  $\mathbb{Z}_2$ -graded,  $\mathfrak{c}$ -invariant (and hence  $(C_0, \mathfrak{c})$ -invariant) subspace of  $\mathcal{H}$ . Since  $(\pi, \rho^\pi, \mathcal{H})$  is irreducible, it follows that  $\mathcal{W} = \mathcal{H}$ . Therefore we have proved that every irreducible unitary representation of  $(C_0, \mathfrak{c})$  is finite dimensional.

Next observe that  $\mathcal{U}(\mathfrak{c})/\langle Z - a \rangle$  is isomorphic (as a  $\mathbb{Z}_2$ -graded algebra) to a complex Clifford algebra. It is a well-known result in the theory of Clifford modules that up to parity change, a complex Clifford algebra has a unique nontrivial finite dimensional irreducible  $\mathbb{Z}_2$ -graded representation. (See [LaMi, Chapter 5] or [CCTV, Lemma 11].) If  $\dim \mathfrak{c}_1$  is odd, then the choice of  $\mathbb{Z}_2$ -grading does not matter, whereas if this dimension is even, then parity change yields two non-isomorphic modules. Conversely, fixing an  $a > 0$  and an irreducible  $\mathbb{Z}_2$ -graded module  $\mathcal{K}$  for the complex Clifford algebra  $\mathcal{U}(\mathfrak{c})/\langle Z - a \rangle$ , one can obtain an irreducible unitary<sup>3</sup> representation  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  of  $(C_0, \mathfrak{c})$ , where  $\mathcal{K}_\mu = \mathcal{K}$  as a vector space and  $\mu : \mathfrak{c}_0 \rightarrow \mathbb{R}$  is an  $\mathbb{R}$ -linear functional such that  $\mu(Z) = a$  and

$$(4.5) \quad \text{for every } W \in \mathfrak{c}_0 \text{ and } v \in \mathcal{K}_\mu, \quad \sigma_\mu^\infty(W)v = \mu(W)\sqrt{-1}v.$$

Note that the condition  $a > 0$  implies that  $\mu([V, V]) > 0$  for every  $V \in \mathfrak{c}_1$ .

In conclusion, if  $(\sigma_0, \rho^{\sigma_0}, \mathcal{K}_0)$  denotes the  $(1|0)$ -dimensional trivial representation of  $(C_0, \mathfrak{c})$ , then we have proved the following statement.<sup>4</sup>

**Proposition 4.5.** *Let  $(C_0, \mathfrak{c})$  be a super Lie group of Clifford type and  $(\pi, \rho^\pi, \mathcal{H})$  be an irreducible unitary representation of  $(C_0, \mathfrak{c})$ . Then there exists a unique  $\mathbb{R}$ -linear functional  $\mu : \mathfrak{c}_0 \rightarrow \mathbb{R}$  satisfying  $\mu([V, V]) \geq 0$  for every  $V \in \mathfrak{c}_1$  such that*

$$(\pi, \rho^\pi, \mathcal{H}) \simeq (\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu).$$

*The representation  $(\pi, \rho^\pi, \mathcal{H})$  is trivial if and only if  $\mu = 0$ .*

## 5. REALIZATION AS INDUCED REPRESENTATIONS

**5.1. Codimension-one induction.** Throughout this section  $(N_0, \mathfrak{n})$  will be a reduced nilpotent super Lie group such that  $\dim \mathfrak{n} > 1$  and  $\dim \mathcal{Z}(\mathfrak{n}) = 1$ . We assume that  $(N_0, \mathfrak{n})$  is not of Clifford type. Hence part (a) of Proposition 4.4 holds for  $\mathfrak{n}$ . Let  $\mathfrak{n}' = \mathfrak{n}'_0 \oplus \mathfrak{n}'_1$ ,  $X$ ,  $Y$ , and  $Z$  be as in part (a) of Proposition 4.4. Let  $(N'_0, \mathfrak{n}')$  be the sub super Lie group of  $(N_0, \mathfrak{n})$  corresponding to  $\mathfrak{n}'$ .

<sup>3</sup>Unitarity of this module follows from standard constructions of Clifford modules. See [LaMi] or [CCTV, Section 4.2].

<sup>4</sup>Strictly speaking, there is ambiguity in the choice of  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  up to parity change. However, for our purposes the choice of  $\mathbb{Z}_2$ -grading does not really matter, since special induction commutes with parity change.

Let  $(\pi, \rho^\pi, \mathcal{H})$  be an irreducible unitary representation of  $(N_0, \mathfrak{n})$ . By [CCTV, Lemma 5], there exists a real number  $b \in \mathbb{R}$  such that for every  $t \in \mathbb{R}$  and  $v \in \mathcal{H}$ , we have

$$\pi(\exp(tZ))v = e^{tb\sqrt{-1}}v.$$

The main goal of this section is to prove the following.

**Proposition 5.1.** *Suppose that the restriction of  $\pi^\infty$  to  $\mathcal{Z}(\mathfrak{n})$  is nontrivial, i.e.,  $b \neq 0$ . Then there exists an irreducible unitary representation  $(\sigma, \rho^\sigma, \mathcal{K})$  of  $(N'_0, \mathfrak{n}')$  such that*

$$\pi \simeq \text{Ind}_{(N'_0, \mathfrak{n}')}^{(N_0, \mathfrak{n})}(\sigma, \rho^\sigma, \mathcal{K}).$$

The rest of this section is devoted to the proof of Proposition 5.1. The proof is inspired by that of [CG, Proposition 2.3.4], but there are several crucial technical points that our proof deviates from the argument given in [CG].

Set  $\mathfrak{h} = \text{Span}_{\mathbb{R}}\{X, Y, Z\}$ . Clearly,  $\mathfrak{h}$  is a Heisenberg Lie subalgebra of  $\mathfrak{n}_0$ , corresponding to a Heisenberg Lie subgroup  $H$  of  $N_0$ .

Fix an  $i \in \{0, 1\}$ . The space  $\mathcal{H}_i$  is an  $N_0$ -invariant subspace of  $\mathcal{H}$ , and we will denote this representation of  $N_0$  by  $(\pi_i, \mathcal{H}_i)$ . Let  $\mathcal{H}_i^\infty$  be the space of smooth vectors of  $(\pi_i, \mathcal{H}_i)$ .

From the proof of [CG, Proposition 2.3.4] it follows that  $\pi_i \simeq \text{Ind}_{N'_0}^{N_0} \sigma_i$  where  $\sigma_i$  is a unitary representation of  $N'_0$ . For the reader's convenience, we give an outline of the argument. By the Stone-von Neumann theorem in the form stated in [CG, 2.2.9], there exist a Hilbert space  $\mathcal{K}_i$  and a linear isometry

$$(5.1) \quad S_i : \mathcal{H}_i \rightarrow L^2(\mathbb{R}, \mathcal{K}_i)$$

such that  $S_i$  intertwines the action of  $H$ , where the action of  $H$  on  $L^2(\mathbb{R}, \mathcal{K}_i)$  is given as follows : for every  $s, t \in \mathbb{R}$  and  $f \in L^2(\mathbb{R}, \mathcal{K}_i)$ ,

$$\begin{aligned} (\pi_i(\exp(tX))f)(s) &= f(s+t) \\ (\pi_i(\exp(tY))f)(s) &= e^{bts\sqrt{-1}}f(s) \\ (\pi_i(\exp(tZ))f)(s) &= e^{tb\sqrt{-1}}f(s). \end{aligned}$$

Lemma 2.3.2 of [CG] is still valid, and Lemma 2.3.1 of [CG] implies that for every  $g \in N'_0$ , there exists a family  $\{T_{g,t}\}_{t \in \mathbb{R}}$  of unitary operators from  $\mathcal{K}_i$  to  $\mathcal{K}_i$  such that for every  $f \in \mathcal{H}_i^\infty$ ,  $g \in N'_0$ , and  $t \in \mathbb{R}$ , we have

$$(\pi_i(g)f)(t) = T_{g,t}(f(t)).$$

The rest of the argument, i.e., showing that the choice of

$$(5.2) \quad \sigma_i(g) = T_{g,0}$$

defines a unitary representation of  $N'_0$  on  $\mathcal{K}_i$ , and that  $\pi_i \simeq \text{Ind}_{N'_0}^{N_0} \sigma_i$ , follows the proof of [CG, Proposition 2.3.4] *mutatis mutandis*.

Since  $\pi_i \simeq \text{Ind}_{N'_0}^{N_0} \sigma_i$ , the Hilbert space  $\mathcal{H}_i$  can be realized as a space of functions from  $N_0$  to  $\mathcal{K}_i$  (see Section 3.1). If we pick this realization of  $\mathcal{H}_i$ , then the isometry  $S_i$  of (5.1) is given by a simple formula which we now describe. Let

$$(5.3) \quad L_0 = \{ \exp(tX) \mid t \in \mathbb{R} \}$$

be the one-parameter subgroup of  $N_0$  corresponding to  $X$ . Then after normalizing the inner products, the map

$$(5.4) \quad S_i : \mathcal{H}_i \rightarrow L^2(\mathbb{R}, \mathcal{K}_i)$$

is given by

$$S_i f(t) = f(\exp(tX)).$$

Let  $\mathcal{H}_i^{\infty, c}$  denote the subspace of  $\mathcal{H}_i^\infty$  consisting of functions with compact support modulo  $N'_0$ . Let  $C^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$  and  $C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$  be the subspaces of  $L^2(\mathbb{R}, \mathcal{K}_i)$  defined by

$$C^{\pi_i}(\mathbb{R}, \mathcal{K}_i) = S_i \mathcal{H}_i^\infty \text{ and } C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i) = S_i \mathcal{H}_i^{\infty, c}.$$

Let  $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$  be the orthogonal direct sum of  $\mathcal{K}_0$  and  $\mathcal{K}_1$  and the isometry

$$S : \mathcal{H}^{\infty, c} \rightarrow L^2(\mathbb{R}, \mathcal{K})$$

be defined as  $S = S_0 \oplus S_1$ . We also set

$$C_c^\pi(\mathbb{R}, \mathcal{K}) = S \mathcal{H}^{\infty, c} \text{ and } C^\pi(\mathbb{R}, \mathcal{K}) = S \mathcal{H}^\infty.$$

As usual, let  $C^\infty(\mathbb{R})$  denote the space of complex valued smooth functions on  $\mathbb{R}$ , and  $C_c^\infty(\mathbb{R})$  denote the subspace of  $C^\infty(\mathbb{R})$  consisting of functions with compact support.

**Lemma 5.2.** *Assume the above notation.*

- (a) *If  $\phi \in C^\infty(\mathbb{R})$  and  $f \in C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$  then  $\phi f \in C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$ .*
- (b) *If  $\phi \in C_c^\infty(\mathbb{R})$  and  $f \in C^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$  then  $\phi f \in C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$ .*

*Proof.* Let  $L_0$  be defined as in (5.3). Every element  $n \in N_0$  can be written uniquely as a product  $n = n' \cdot l$  of an element  $n' \in N'_0$  and an element  $l \in L_0$ . Consider the function  $\psi : N_0 \rightarrow \mathbb{C}$  defined by  $\psi(n) = \phi(l)$ . It is easily seen that  $\psi$  is smooth. To prove part (a) it suffices to show that if  $h \in \mathcal{H}_i^{\infty, c}$ , then  $\psi h \in \mathcal{H}_i^{\infty, c}$ , and the latter inclusion follows from the description of smooth vectors for induced representations in [Po, Theorem 5.1] or [CG, Theorem A.1.4]. The proof of part (b) is similar. □

**Lemma 5.3.** *Let  $V \in \mathfrak{n}_1$ . If  $\phi \in C_c^\infty(\mathbb{R})$  and  $f \in C^\pi(\mathbb{R}, \mathcal{K})$  then*

$$\rho^\pi(V)(\phi f) = \phi \rho^\pi(V)f.$$

*Proof.* Let  $M_{\chi_a} : L^2(\mathbb{R}, \mathcal{K}) \rightarrow L^2(\mathbb{R}, \mathcal{K})$  be the operator of multiplication by  $\chi_a$ , i.e.,

$$\text{for every } h \in L^2(\mathbb{R}, \mathcal{K}) \text{ and } t \in \mathbb{R}, \quad (M_{\chi_a} h)(t) = \chi_a(t)h(t),$$

where  $\chi_a(t) = e^{at\sqrt{-1}}$ . By Lemma 5.2, for every  $i \in \{0, 1\}$  we have

$$M_{\chi_a}(C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i)) \subseteq C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i).$$

From  $[Y, V] = 0$  it follows that for every  $a \in \mathbb{R}$ ,

$$(5.5) \quad \rho^\pi(V)M_{\chi_a} = M_{\chi_a}\rho^\pi(V).$$

Choose a sequence  $\{\phi_n\}_{n=1}^\infty$  of elements of  $\text{Span}_{\mathbb{C}}\{\chi_a \mid a \in \mathbb{R}\}$  such that

$$(5.6) \quad \lim_{n \rightarrow \infty} \phi_n f = \phi f \quad \text{and} \quad \lim_{n \rightarrow \infty} \phi_n \rho^\pi(V) f = \phi \rho^\pi(V) f,$$

where the convergences are in  $L^2(\mathbb{R}, \mathcal{K})$ . The sequence  $\{\phi_n\}_{n=1}^\infty$  can be found as follows. By the Stone-Weierstrass theorem, for every positive integer  $n$  one can choose a function

$$\phi_n \in \text{Span}_{\mathbb{C}}\{\chi_a \mid a \in \mathbb{R}\}$$

which is periodic with period  $2n$ , and

$$\max_{-n \leq t \leq n} \{|\phi(t) - \phi_n(t)|\} \leq \frac{1}{n}.$$

Since elements of  $C^\pi(\mathbb{R}, \mathcal{K})$  are smooth vectors for the action of the Heisenberg group  $H$ , they are Schwartz functions from  $\mathbb{R}$  to  $\mathcal{K}$ . Suppose  $n$  is large enough such that

$$\text{Supp}(\phi) \subset \{x \in \mathbb{R} \mid -n \leq x \leq n\}.$$

Now we have

$$\begin{aligned} \|(\phi - \phi_n)f\|^2 &\leq \int_{-n}^n \|(\phi_n(t) - \phi(t))f(t)\|^2 dt + \int_{|t|>n} \|(\phi_n(t) - \phi(t))f(t)\|^2 dt \\ &\leq \frac{1}{n^2} \times 2n \times \max_{|t| \leq n} \{\|f(t)\|^2\} + \left(\frac{1}{n} + \max_{t \in \mathbb{R}} \{|\phi(t)|\}\right)^2 \int_{|t|>n} \|f(t)\|^2 dt. \end{aligned}$$

Since  $f \in C^\pi(\mathbb{R}, \mathcal{K})$ , when  $n$  grows to infinity the last line above converges to zero. Since  $\rho^\pi(V)f \in C^\pi(\mathbb{R}, \mathcal{K})$ , the same reasoning applies to  $\rho^\pi(V)f$  instead of  $f$  as well.

Since  $\phi_n \in \text{Span}_{\mathbb{C}}\{\chi_a \mid a \in \mathbb{R}\}$ , it follows from (5.5) that

$$\rho^\pi(V)\phi_n f = \phi_n \rho^\pi(V)f.$$

The operator  $\rho^\pi(V)$  is symmetric, hence it is closable (see [Co, p. 316]). In particular, from (5.6) and the fact that  $\phi f \in D(\rho^\pi(V))$  it follows that

$$\rho^\pi(V)(\phi f) = \phi \rho^\pi(V)f.$$

□

Consider the map  $\Psi : N'_0 \times \mathbb{R} \rightarrow N_0$  defined as

$$(5.7) \quad \Psi(n', s) = n' \cdot \exp(sX).$$

The map  $\Psi$  is bijective and the Campbell-Baker-Hausdorff formula implies that it is smooth. Moreover, for every  $\mathbf{x} = (n', s) \in N'_0 \times \mathbb{R}$ , if by means

of the left action of  $N_0$  we identify the tangent spaces at  $\mathbf{x}$  and  $\Psi(\mathbf{x})$  with  $\mathfrak{n}'_0 \oplus \mathbb{R}$  and  $\mathfrak{n}_0$ , then the derivative

$$\mathbf{D}\Psi(\mathbf{x}) : \mathfrak{n}'_0 \oplus \mathbb{R} \rightarrow \mathfrak{n}_0$$

of  $\Psi$  at  $\mathbf{x}$  is given by the following formula.

$$\text{For every } (Q, t) \in \mathfrak{n}'_0 \oplus \mathbb{R}, \quad \mathbf{D}\Psi(\mathbf{x})(Q, t) = (\exp(-sX) \cdot Q, t).$$

Note that  $\exp(-sX) \cdot Q \in \mathfrak{n}'_0$  because  $N'_0$  is a normal subgroup of  $N_0$  (see [CG, Lemma 1.1.8]).

From the formula for  $\mathbf{D}\Psi$  it follows immediately that  $\mathbf{D}\Psi(\mathbf{x})$  is invertible at every  $\mathbf{x} \in N'_0 \times \mathbb{R}$ . Hence the inverse mapping theorem implies that  $\Psi^{-1}$  is smooth. Consequently, a function  $f : N_0 \rightarrow \mathcal{K}_i$  is smooth if and only if  $f \circ \Psi : N'_0 \times \mathbb{R} \rightarrow \mathcal{K}_i$  is smooth.

**Lemma 5.4.** *Let  $f : \mathbb{R} \rightarrow \mathcal{K}_i$  be a smooth function such that  $f(0) = 0$ . Set*

$$g(t) = \begin{cases} \frac{f(t)}{t} & \text{if } t \neq 0, \\ f'(0) & \text{otherwise.} \end{cases}$$

*Then  $g$  is a smooth function as well, and  $g^{(n)}(0) = \frac{f^{(n+1)}(0)}{n+1}$ .*

*Proof.* For  $t \neq 0$ , the lemma is trivial. We prove the Lemma for  $t = 0$  by induction on  $n$ , in each step proving that  $g^{(n)}(0) = \frac{f^{(n+1)}(0)}{n+1}$ . For  $n = 0$  the latter statement is obvious. We will assume that the statement is true for some  $n$ , and prove it for  $n + 1$ . If  $t \neq 0$ , then

$$g^{(n)}(t) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (n-k)! t^{-n+k-1} f^{(k)}(t).$$

Therefore

$$\lim_{t \rightarrow 0} \frac{g^{(n)}(t) - g^{(n)}(0)}{t} = \lim_{t \rightarrow 0} \frac{\left( \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (n-k)! t^k f^{(k)}(t) \right) - \frac{t^{n+1} f^{(n+1)}(0)}{n+1}}{t^{n+2}}$$

which is a limit of the form

$$\lim_{t \rightarrow 0} \frac{h_1(t)}{h_2(t)}$$

where  $h_1$  and  $h_2$  are continuously differentiable functions and  $h_1(0) = h_2(0) = 0$ . It follows that the above limit is equal to

$$\lim_{t \rightarrow 0} \frac{h'_1(t)}{h'_2(t)}$$

in case the latter limit exists. But we have

$$\begin{aligned}
h'_1(t) &= n! \sum_{k=0}^n (-1)^{n-k} f^{(k+1)}(t) \frac{t^k}{k!} \\
&+ \left( n! \sum_{k=1}^n (-1)^{n-k} f^{(k)}(t) \frac{t^{k-1}}{(k-1)!} \right) - t^n f^{(n+1)}(0) \\
&= t^n f^{(n+1)}(t) - t^n f^{(n+1)}(0)
\end{aligned}$$

while  $h'_2(t) = (n+2)t^{n+1}$ . In conclusion, we have

$$\lim_{t \rightarrow 0} \frac{h'_1(t)}{h'_2(t)} = \lim_{t \rightarrow 0} \frac{(t^n f^{(n+1)}(t) - t^n f^{(n+1)}(0))}{(n+2)t^{n+1}} = \frac{f^{(n+2)}(0)}{n+2}$$

which implies that  $g^{(n+1)}(0) = \frac{f^{(n+2)}(0)}{n+2}$ .  $\square$

**Lemma 5.5.** *Let  $q$  be a positive integer and  $f : \mathbb{R}^q \times \mathbb{R} \rightarrow \mathcal{K}_i$  be a smooth function such that for every  $x \in \mathbb{R}^q$ , we have  $f(x, 0) = 0$ . Define*

$$g(x, t) = \begin{cases} \frac{f(x, t)}{t} & \text{if } t \neq 0, \\ \frac{\partial f}{\partial t}(x, 0) & \text{otherwise.} \end{cases}$$

Then  $g(x, t)$  is smooth, and indeed

$$(5.8) \quad \frac{\partial^n g}{\partial t^n}(x, 0) = \frac{\frac{\partial^{n+1} f}{\partial t^{n+1}}(x, 0)}{n+1}.$$

*Proof.* That  $g(x, t)$  is smooth when  $t \neq 0$  is trivial. From Lemma 5.4 it follows that for every integer  $n \geq 0$  and every  $x \in \mathbb{R}^q$ ,  $\frac{\partial^n g}{\partial t^n}(x, 0)$  exists and equality (5.8) holds.

Every differential operator in  $x_1, \dots, x_q, t$  is a linear combination of operators  $\mathcal{D}$  of the form

$$\mathcal{D} = \frac{\partial^{a_1}}{\partial x_{i_1}^{a_1}} \frac{\partial^{b_1}}{\partial t^{b_1}} \frac{\partial^{a_2}}{\partial x_{i_2}^{a_2}} \frac{\partial^{b_2}}{\partial t^{b_2}} \cdots \frac{\partial^{a_k}}{\partial x_{i_k}^{a_k}} \frac{\partial^{b_k}}{\partial t^{b_k}}$$

where  $i_1, \dots, i_k \in \{1, \dots, q\}$  and  $a_1, \dots, a_k, b_1, \dots, b_k \in \{0, 1\}$ . (For example, if  $k = 3$ ,  $a_1 = a_3 = 1$ ,  $a_2 = 0$ ,  $b_1 = b_2 = 1$ ,  $b_3 = 0$ ,  $i_1 = 3$ ,  $i_3 = 2$ , and  $1 \leq i_2 \leq q$  then  $\mathcal{D} = \frac{\partial}{\partial x_3} \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial x_2}$ .) In order to complete the proof of the lemma, it suffices to show that for every such  $\mathcal{D}$  and every  $x \in \mathbb{R}^q$ , the partial derivative  $\mathcal{D}g(x, 0)$  exists.

For every  $1 \leq i \leq q$  and  $n \geq 0$  we have

$$\frac{\partial}{\partial x_i} \frac{\partial^n g}{\partial t^n}(x, t) = \begin{cases} \frac{\partial^n}{\partial t^n} \left( \frac{1}{t} \frac{\partial f}{\partial x_i} \right)(x, t) & \text{if } t \neq 0, \\ \frac{1}{n+1} \left( \frac{\partial^{n+1} f}{\partial t^{n+1}} \frac{\partial f}{\partial x_i} \right)(x, 0) & \text{otherwise.} \end{cases}$$

Thus for every  $(x, t) \in \mathbb{R}^q \times \mathbb{R}$  the partial derivative  $\frac{\partial}{\partial x_i} \frac{\partial^n g}{\partial t^n}(x, t)$  exists and if we set

$$g_1(x, t) = \begin{cases} \frac{\frac{\partial f}{\partial x_i}(x, t)}{t} & \text{if } t \neq 0, \\ \frac{\partial}{\partial t} \frac{\partial f}{\partial x_i}(x, 0) & \text{otherwise} \end{cases}$$

then we have

$$\frac{\partial}{\partial x_i} \frac{\partial^n g}{\partial t^n}(x, t) = \frac{\partial^n g_1}{\partial t^n}(x, t).$$

Note that Lemma 5.4 implies that  $\frac{\partial^n g_1}{\partial t^n}(x, t)$  exists for every  $n \geq 0$ .

By repeating the above argument one can show that  $\mathcal{D}g(x, t)$  exists and is equal to

$$\frac{\partial^{b_1+\dots+b_k}}{\partial t^{b_1+\dots+b_k}} g_2(x, t)$$

where

$$g_2(x, t) = \begin{cases} \frac{1}{t} \left( \frac{\partial^{a_1+\dots+a_k}}{\partial x_{i_1}^{a_1} \partial x_{i_2}^{a_2} \dots \partial x_{i_k}^{a_k}} f(x, t) \right) & \text{if } t \neq 0, \\ \frac{\partial}{\partial t} \left( \frac{\partial^{a_1+\dots+a_k}}{\partial x_{i_1}^{a_1} \partial x_{i_2}^{a_2} \dots \partial x_{i_k}^{a_k}} f \right)(x, 0) & \text{otherwise.} \end{cases}$$

The existence of  $\frac{\partial^{b_1+\dots+b_k}}{\partial t^{b_1+\dots+b_k}} g_2(x, t)$  follows from Lemma 5.4. □

**Lemma 5.6.** *Let  $f \in C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$  satisfy  $f(0) = 0$ . Then there exists a function  $g \in C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$  such that for every  $t \in \mathbb{R}$  we have  $f(t) = tg(t)$ .*

*Proof.* Let  $f = S_i h$  where  $S_i$  is the operator defined in (5.4) and  $h \in \mathcal{H}_i^{\infty, c}$ . Set  $h_1 = h \circ \Psi$  where  $\Psi : N'_0 \times \mathbb{R} \rightarrow N_0$  is the map defined in (5.7). For every  $(n', t) \in N'_0 \times \mathbb{R}$  we have

$$h_1(n', t) = t h_2(n', t)$$

where  $h_2 : N'_0 \times \mathbb{R} \rightarrow \mathcal{K}_i$  is defined as follows.

$$h_2(n', t) = \begin{cases} \frac{h_1(n', t)}{t} & \text{if } t \neq 0, \\ \frac{\partial h_1}{\partial t}(n', 0) & \text{otherwise.} \end{cases}$$

Lemma 5.5 implies that  $h_2$  is smooth. From the description of smooth vectors for induced representations given in [Po, Theorem 5.1] or [CG, Theorem A.1.4] it follows that the function  $h_3 = h_2 \circ \Psi^{-1}$  belongs to  $\mathcal{H}_i^{\infty, c}$ . To complete the proof, we set  $g = S_i h_3$ . □

**Lemma 5.7.** *Let  $V \in \mathfrak{n}_1$ . Suppose that  $f \in C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$  satisfies  $f(0) = 0$ . Then*

$$(\rho^\pi(V)f)(0) = 0.$$

*Proof.* By Lemma 5.6 we have  $f(t) = tg(t)$  where  $g \in C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$ . Since  $g$  has compact support, it is not hard to see that there exists a function  $\psi \in C_c^\infty(\mathbb{R})$  such that  $\psi(t) = 1$  for  $t \in \text{Supp}(g) \cup \{0\}$ . It follows that  $f = \psi f$ . Set  $\psi_1(t) = t\psi(t)$ . By Lemma 5.3 we have

$$Tf = T\psi f = T(\psi_1 g) = \psi_1 Tg,$$

hence  $(Tf)(0) = \psi_1(0)((Tg)(0)) = 0$ , which completes the proof.  $\square$

**Remark.** 1. Let  $V \in \mathfrak{n}_1$  and suppose that  $f \in C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$  satisfies  $f(t_o) = 0$  for some  $t_o \in \mathbb{R}$ . Then  $(\pi_i(\exp(t_o X))f)(0) = 0$ , hence by Lemma 5.7

$$\begin{aligned} (\rho^{\pi_i}(V)f)(t_o) &= (\pi_i(\exp(t_o X))\rho^{\pi_i}(V)f)(0) \\ (5.9) \quad &= \left( \rho^{\pi_i}(\exp(t_o X) \cdot V) \pi_i(\exp(t_o X))f \right)(0) = 0 \end{aligned}$$

2. An immediate consequence of (5.9) is that  $\rho^\pi(V)C_c^\pi(\mathbb{R}, \mathcal{K}) \subseteq C_c^\pi(\mathbb{R}, \mathcal{K})$ .

For every  $V \in \mathfrak{n}_1$  we define a family of linear operators

$$T_{V,t} : \mathcal{K}_1^\infty \oplus \mathcal{K}_2^\infty \rightarrow \mathcal{K}_1^\infty \oplus \mathcal{K}_2^\infty$$

as follows. For every  $i \in \{0, 1\}$ ,  $t_o \in \mathbb{R}$ , and  $v \in \mathcal{K}_i^\infty$ , choose an  $f \in C_c^{\pi_i}(\mathbb{R}, \mathcal{K}_i)$  such that  $f(t_o) = v$ . For instance, one can fix  $\varphi \in C_c^\infty(\mathbb{R})$  such that  $\varphi(t_o) = 1$ , and take  $f = S_i(h \circ \Psi^{-1})$  where  $h : N'_0 \times \mathbb{R} \rightarrow \mathcal{K}_i$  is given by

$$h(n', t) = \varphi(t)\sigma_i(n')v.$$

Now set

$$T_{V,t}v = (\rho^\pi(V)f)(t).$$

Lemma 5.7 and the remark after this lemma imply that the operators  $T_{V,t}$  are well defined. Since  $\rho^\pi(V)$  is odd, it follows that the  $T_{V,t}$ 's are odd operators.

We now set  $\sigma = \sigma_0 \oplus \sigma_1$ , and for any  $W \in \mathfrak{n}_1$  define  $\rho^\sigma(W) : \mathcal{K}^\infty \rightarrow \mathcal{K}^\infty$  by  $\rho^\sigma(W)v = T_{W,0}v$ . Our next task is to verify that the triple  $(\sigma, \rho^\sigma, \mathcal{K})$  satisfies the conditions of Definition 2.2. Linearity of  $\rho^\sigma$  and condition (a) of Definition 2.2 are obvious. Next we prove that for every  $W \in \mathfrak{n}_1$  the operator  $\rho^\sigma(W)$  is symmetric. Suppose, on the contrary, that  $\rho^\sigma(W)$  is not symmetric, and let  $v, w \in \mathcal{K}^\infty$  such that

$$\langle \rho^\sigma(W)v, w \rangle \neq \langle v, \rho^\sigma(W)w \rangle.$$

Choose  $\varphi \in C_c^\infty(\mathbb{R})$  such that  $\varphi(0) = 1$ , and consider two functions

$$f_v, f_w : N'_0 \times \mathbb{R} \rightarrow \mathcal{K}$$

defined by

$$f_v(n', t) = \varphi(t)\sigma(n')v \quad \text{and} \quad f_w(n', t) = \varphi(t)\sigma(n')w.$$

The functions  $f_v \circ \Psi^{-1}$  and  $f_w \circ \Psi^{-1}$  belong to  $\mathcal{H}^{\infty, c}$ . Let  $g_v, g_w \in C_c^\pi(\mathbb{R}, \mathcal{K})$  be defined by

$$g_v = S(f_v \circ \Psi^{-1}) \quad \text{and} \quad g_w = S(f_w \circ \Psi^{-1}).$$

It is readily seen that

$$(\rho^\pi(W)g_v)(t) = T_{\exp(tX) \cdot W, 0}(g_v(t)).$$

If  $\{W, W_1, \dots, W_r\}$  is a basis containing  $W$  for  $\mathfrak{n}_1$ , then

$$(5.10) \quad \exp(tX) \cdot W = \gamma_0(t)W + \sum_{i=1}^r \gamma_i(t)W_i$$

where for every  $0 \leq i \leq r$  the function  $\gamma_i(t) : \mathbb{R} \rightarrow \mathbb{R}$  is smooth. Moreover,

$$\lim_{t \rightarrow 0} \gamma_0(t) = 1 \quad \text{and for every } 1 \leq i \leq r, \quad \lim_{t \rightarrow 0} \gamma_i(t) = 0.$$

Now

$$\begin{aligned} (\rho^\pi(W)g_v)(t) &= T_{\exp(tX) \cdot W, 0}(g_v(t)) \\ &= \gamma_0(t)T_{W, 0}(g_v(t)) + \sum_{i=1}^r \gamma_i(t)T_{W_i, 0}(g_v(t)) \\ &= \varphi(t) \left( \gamma_0(t)T_{W, 0}v + \sum_{i=1}^r \gamma_i(t)T_{W_i, 0}v \right). \end{aligned}$$

But

$$(5.11) \quad \lim_{t \rightarrow 0} \left( \gamma_0(t)T_{W, 0}v + \sum_{i=1}^r \gamma_i(t)T_{W_i, 0}v \right) = T_{W, 0}v = \rho^\sigma(W)v$$

and

$$\begin{aligned} (5.12) \quad \langle \rho^\pi(W)g_v, g_w \rangle &= \int_{-\infty}^{\infty} \langle (\rho^\pi(W)g_v)(t), g_w(t) \rangle dt \\ &= \int_{-\infty}^{\infty} \varphi(t)^2 \langle \left( \gamma_0(t)T_{W, 0}v + \sum_{i=1}^r \gamma_i(t)T_{W_i, 0}v \right), w \rangle dt. \end{aligned}$$

Similarly we have

$$(5.13) \quad \lim_{t \rightarrow 0} \left( \gamma_0(t)T_{W, 0}w + \sum_{i=1}^r \gamma_i(t)T_{W_i, 0}w \right) = T_{W, 0}w = \rho^\sigma(W)w$$

and

$$\begin{aligned} (5.14) \quad \langle \rho^\pi(W)g_w, g_v \rangle &= \int_{-\infty}^{\infty} \langle g_v(t), (\rho^\pi(W)g_w)(t) \rangle dt \\ &= \int_{-\infty}^{\infty} \varphi(t)^2 \langle v, \left( \gamma_0(t)T_{W, 0}w + \sum_{i=1}^r \gamma_i(t)T_{W_i, 0}w \right) \rangle dt. \end{aligned}$$

From (5.11), (5.12), (5.13) and (5.14) it follows that if  $\text{Supp}(\varphi)$  is small enough, then

$$\langle \rho^\pi(W)g_v, g_w \rangle \neq \langle g_v, \rho^\pi(W)g_w \rangle$$

which contradicts the fact that  $\rho^\pi(W)$  is symmetric.

Condition (c) of Definition 2.2 can be verified as follows. Let  $V, W \in \mathfrak{g}_1$  and  $v \in \mathcal{K}^\infty$ . Choose an  $f \in C_c^\pi(\mathbb{R}, \mathcal{K})$  such that  $f(0) = v$ . We have

$$\begin{aligned} (\rho^\sigma(V)\rho^\sigma(W) + \rho^\sigma(W)\rho^\sigma(V))v &= \left( \rho^\pi(V)(\rho^\pi(W)f) \right)(0) + \left( \rho^\pi(W)(\rho^\pi(V)f) \right)(0) \\ &= -\sqrt{-1}(\pi^\infty([V, W])f)(0). \end{aligned}$$

Since  $[V, W] \in \mathfrak{n}'_0$ , from (5.2) it follows that

$$(\pi^\infty([V, W]))f(0) = \sigma^\infty([V, W])(f(0)) = \sigma^\infty([V, W])v$$

which completes the proof of condition (c).

Finally, we prove condition (d) of Definition 2.2. Let  $V \in \mathfrak{n}_1$ ,  $g \in N'_0$ , and  $w \in \mathcal{K}^\infty$ . Let  $f \in C_c^\pi(\mathbb{R}, \mathcal{K})$  be such that  $f(0) = w$ . Using (5.2) we have

$$\begin{aligned} \rho^\sigma(g \cdot V)w &= (\rho^\pi(g \cdot V)f)(0) \\ &= (\pi(g)\rho^\pi(V)\pi(g)^{-1}f)(0) \\ &= \sigma(g)\left((\rho^\pi(V)\pi(g^{-1})f)(0)\right) \\ &= \sigma(g)\left(\rho^\sigma(V)\left((\pi(g^{-1})f)(0)\right)\right) \\ &= \sigma(g)\left(\rho^\sigma(V)\left(\sigma(g^{-1})(f(0))\right)\right) \\ &= \sigma(g)\rho^\sigma(V)\sigma(g^{-1})w. \end{aligned}$$

To finish the proof of Proposition 5.1, note that the unitary representations  $(\pi, \rho^\pi, \mathcal{H})$  and

$$\text{Ind}_{(N'_0, \mathfrak{n}')}^{(N_0, \mathfrak{n})}(\sigma, \rho^\sigma, \mathcal{K})$$

are identical on  $\mathcal{H}^{\infty, c}$ . This follows from the fact that for every  $V \in \mathfrak{n}_1$ ,  $t \in \mathbb{R}$ , and  $f \in C_c^\pi(\mathbb{R}, \mathcal{K})$  we have

$$(\rho^\pi(V)f)(t) = \rho^\sigma(\exp(tX) \cdot V)(f(t)).$$

Consequently, Proposition 2.3 implies that these representations are unitarily equivalent. Since  $(\pi, \rho^\pi, \mathcal{H})$  is assumed to be irreducible, it follows that  $(\sigma, \rho^\sigma, \mathcal{K})$  is irreducible as well.

**5.2. Stone-von Neumann theorem for Heisenberg-Clifford supergroups.** In this section we show how to use Proposition 5.1 to prove a generalization of the Stone-von Neumann theorem for Heisenberg-Clifford super Lie groups.

Let  $(N_0, \mathfrak{n})$  be a Heisenberg-Clifford super Lie group (see Section 4.2). For every  $P, Q \in \mathfrak{n}_1$ , the value of the bracket  $[P, Q]$  lies in  $\mathfrak{n}_0 = \mathbb{R}$ , and hence can be thought of as a real number. Consider the symmetric bilinear form

$$B : \mathfrak{n}_1 \times \mathfrak{n}_1 \rightarrow \mathbb{R}$$

defined by  $B(P, Q) = [P, Q]$ .

Let  $(\pi, \rho^\pi, \mathcal{H})$  be an irreducible unitary representation of  $(N_0, \mathfrak{n})$ . By [CCTV, Lemma 5] the action of  $\mathcal{Z}(N_0)$  is via multiplication by a unitary character

$$\chi : \mathcal{Z}(N_0) \rightarrow \mathbb{C}^\times.$$

When  $B$  is a definite form, we say that the character  $\chi$  *agrees with*  $B$  if there exists a positive real number  $c$  such that for every  $P \in \mathfrak{n}_1$ ,

$$\chi([P, P]) = e^{cB(P, P)\sqrt{-1}}.$$

If  $\chi$  is the trivial character, then it is easily seen that the sub super Lie group  $(\mathcal{Z}(N_0), [\mathfrak{n}_1, \mathfrak{n}_1] \oplus \mathfrak{n}_1)$  of  $(N_0, \mathfrak{n})$  belongs to the kernel of  $(\pi, \rho^\pi, \mathcal{H})$ . Consequently,  $(\pi, \rho^\pi, \mathcal{H})$  yields an irreducible representation of the abelian Lie group  $N_0/\mathcal{Z}(N_0)$ . It follows that  $(\pi, \rho^\pi, \mathcal{H})$  is a one-dimensional representation obtained from a unitary character of  $N_0/\mathcal{Z}(N_0)$ .

If  $\chi$  is not trivial, then we have the following result.

**Theorem 5.8.** *Suppose that the unitary character  $\chi : \mathcal{Z}(N_0) \rightarrow \mathbb{C}^\times$  is nontrivial.*

- (a) *If  $B$  is an indefinite form, then there are no irreducible unitary representations of  $(N_0, \mathfrak{n})$  with central character  $\chi$ .*
- (b) *Suppose that  $B$  is a definite form. If  $\chi$  agrees with  $B$ , then up to unitary equivalence and parity change there exists a unique irreducible unitary representation of  $(N_0, \mathfrak{n})$  with central character  $\chi$ . If  $\chi$  does not agree with  $B$ , then such a unitary representation does not exist.*

*Proof.* Part (a) follows from the fact that  $\mathfrak{a}[\mathfrak{n}] = [\mathfrak{n}_1, \mathfrak{n}_1] \oplus \mathfrak{n}_1$ . Part (b) is proved by induction on the dimension of  $N_0$  as follows. Let

$$\{Z, X_1, X_2, \dots, X_m, Y_1, \dots, Y_m, V_1, \dots, V_n\}$$

be the basis of  $\mathfrak{n}$  given in (4.1), and  $(\pi, \rho^\pi, \mathcal{H})$  be an irreducible unitary representation of  $(N_0, \mathfrak{n})$  with central character  $\chi$ . By Proposition 5.1 we have

$$(\pi, \rho^\pi, \mathcal{H}) = \text{Ind}_{(N'_0, \mathfrak{n}')}^{(N_0, \mathfrak{n})}(\sigma, \rho^\sigma, \mathcal{K})$$

where

$$\mathfrak{n}' = \text{Span}_{\mathbb{R}}\{Z, X_2, \dots, X_m, Y_1, \dots, Y_m, V_1, \dots, V_n\}.$$

Moreover, from the proof of Proposition 5.1 it follows that  $\sigma^\infty(Y_1) = 0$ . Therefore  $(\sigma, \rho^\sigma, \mathcal{K})$  factors through a representation of a Heisenberg-Clifford super Lie group  $(N''_0, \mathfrak{n}'')$  where  $\mathfrak{n}'' = \mathfrak{n}'/\mathfrak{r}$  and  $\mathfrak{r} = \text{Span}_{\mathbb{R}}\{Y_1\}$ . Since  $\dim N''_0 < \dim N_0$ , the proof is completed by induction on  $\dim N_0$ . Details are left to the reader. □

**Remark.** Suppose that  $\chi$  is nontrivial,  $B$  is definite, and  $\chi$  agrees with  $B$ . Then part (b) of Theorem 5.8 can be refined slightly as follows. When  $\dim \mathfrak{n}_1$  is even, there exist two irreducible unitary representations which are not unitarily equivalent. However, when  $\dim \mathfrak{n}_1$  is odd, we obtain a unique

such representation up to unitary equivalence. Indeed the restriction to  $(\mathbb{Z}(N_0), [\mathfrak{n}_1, \mathfrak{n}_1] \oplus \mathfrak{n}_1)$  of such a representation is a countable direct sum of modules for a complex Clifford algebra, and when  $\dim \mathfrak{n}_1$  is even there are two nonisomorphic such modules [LaMi, Chapter 5]. The details are left to the reader.

## 6. POLARIZING SYSTEMS AND MAIN THEOREMS

**6.1. Polarizing systems.** Throughout this section  $(N_0, \mathfrak{n})$  is a (not necessarily reduced) nilpotent super Lie group.

**Definition 6.1.** *A polarizing system of  $(N_0, \mathfrak{n})$  is a 6-tuple  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  where*

- (a)  $(M_0, \mathfrak{m})$  is a special sub super Lie group of  $(N_0, \mathfrak{n})$ .
- (b)  $\lambda : \mathfrak{n}_0 \rightarrow \mathbb{R}$  is an  $\mathbb{R}$ -linear functional and  $\mathfrak{m}_0$  is a polarizing subalgebra of  $\mathfrak{n}_0$  corresponding to  $\lambda$ .
- (c)  $(C_0, \mathfrak{c})$  is a super Lie group of Clifford type and  $\Phi$  is a surjective homomorphism

$$\Phi : (M_0, \mathfrak{m}) \rightarrow (C_0, \mathfrak{c}).$$

- (d)  $\mathfrak{m}_0 \cap \ker \Phi = \mathfrak{m}_0 \cap \ker \lambda$ .

Let  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  be a polarizing system of  $(N_0, \mathfrak{n})$  and  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  be the irreducible unitary representation of  $(C_0, \mathfrak{c})$  associated to a linear functional  $\mu : \mathfrak{c}_0 \rightarrow \mathbb{R}$  (see Section 4.5). One can compose  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  with the map

$$\Phi : (M_0, \mathfrak{m}) \rightarrow (C_0, \mathfrak{c})$$

and obtain an irreducible unitary representation  $(\sigma_\mu \circ \Phi, \rho^{\sigma_\mu \circ \Phi}, \mathcal{K}_\mu)$  of  $(M_0, \mathfrak{m})$ . The representation  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  is said to be *consistent* with the polarizing system if

$$(6.1) \quad \text{for every } W \in \mathfrak{m}_0, \quad \lambda(W) = \mu \circ \Phi(W).$$

We will see below that consistent representations play a special role in the classification of irreducible unitary representations.

**Theorem 6.2.** *Let  $(\pi, \rho^\pi, \mathcal{H})$  be an irreducible unitary representation of a nilpotent super Lie group  $(N_0, \mathfrak{n})$ .*

- (a) *There exists a polarizing system  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  and an irreducible unitary representation  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  of  $(C_0, \mathfrak{c})$  which is consistent with*

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$$

*such that*

$$(6.2) \quad (\pi, \rho^\pi, \mathcal{H}) = \text{Ind}_{(M_0, \mathfrak{m})}^{(N_0, \mathfrak{n})} (\sigma_\mu \circ \Phi, \rho^{\sigma_\mu \circ \Phi}, \mathcal{K}_\mu).$$

(b) Suppose that  $(M'_0, \mathfrak{m}', \Phi', C'_0, \mathfrak{c}', \lambda')$  is another polarizing system and

$$(\sigma_{\mu'}, \rho^{\sigma_{\mu'}}, \mathcal{K}_{\mu'})$$

is a representation of  $(C'_0, \mathfrak{c}')$  consistent with  $(M'_0, \mathfrak{m}', \Phi', C'_0, \mathfrak{c}', \lambda')$  such that

$$(\pi, \rho^\pi, \mathcal{H}) = \text{Ind}_{(M'_0, \mathfrak{m}')}^{(N_0, \mathfrak{n})}(\sigma_{\mu'} \circ \Phi, \rho^{\sigma_{\mu'} \circ \Phi}, \mathcal{K}_{\mu'}).$$

Then there exists an  $n \in N_0$  such that

$$\lambda' = \text{Ad}^*(n)(\lambda).$$

Moreover, the super Lie groups  $(C_0, \mathfrak{c})$  and  $(C'_0, \mathfrak{c}')$  are isomorphic.

*Proof.* Part (a) is proved by induction on  $\dim \mathfrak{n}$ . There are three cases to consider:

*Case I :*  $(N_0, \mathfrak{n})$  is not reduced. In this case  $(\pi, \rho^\pi, \mathcal{H})$  factors through the reduced form  $(\overline{N}_0, \overline{\mathfrak{n}})$  of  $(N_0, \mathfrak{n})$ , and  $\dim \overline{\mathfrak{n}} < \dim \mathfrak{n}$ . Let us denote this representation of  $(\overline{N}_0, \overline{\mathfrak{n}})$  by  $(\overline{\pi}, \overline{\rho}^\pi, \mathcal{H})$ . By induction hypothesis, there exists a polarizing system  $(\overline{M}_0, \overline{\mathfrak{m}}, \overline{\Phi}, \overline{C}_0, \overline{\mathfrak{c}}, \overline{\lambda})$  of  $(\overline{N}_0, \overline{\mathfrak{n}})$  and a representation  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  of  $(\overline{C}_0, \overline{\mathfrak{c}})$  which is consistent with  $(\overline{M}_0, \overline{\mathfrak{m}}, \overline{\Phi}, \overline{C}_0, \overline{\mathfrak{c}}, \overline{\lambda})$  such that

$$(\overline{\pi}, \overline{\rho}^\pi, \mathcal{H}) = \text{Ind}_{(\overline{M}_0, \overline{\mathfrak{m}})}^{(\overline{N}_0, \overline{\mathfrak{n}})}(\sigma_\mu \circ \overline{\Phi}, \rho^{\sigma_\mu \circ \overline{\Phi}}, \mathcal{K}_\mu).$$

Let  $q : (N_0, \mathfrak{n}) \rightarrow (\overline{N}_0, \overline{\mathfrak{n}})$  be the quotient map and set

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda) = (q^{-1}(\overline{M}_0), q^{-1}(\overline{\mathfrak{m}}), \overline{\Phi} \circ q, \overline{C}_0, \overline{\mathfrak{c}}, \overline{\lambda} \circ q).$$

It is easily checked that  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  is a polarizing system of  $(N_0, \mathfrak{n})$ , and  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  is consistent with  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$ .

*Case II :*  $(N_0, \mathfrak{n})$  is reduced and  $\dim \mathcal{Z}(\mathfrak{n}) > 1$ . Since the action of  $\mathcal{Z}(\mathfrak{n})$  is via scalar multiplication [CCTV, Lemma 5], it is easily seen that  $(\pi, \rho^\pi, \mathcal{H})$  factors through a representation of a quotient  $(N'_0, \mathfrak{n}')$  of  $(N_0, \mathfrak{n})$  where the kernel of the quotient corresponds to a subalgebra of codimension one in  $\mathcal{Z}(\mathfrak{n})$ . Again  $\dim \mathfrak{n}' < \dim \mathfrak{n}$ , and an argument similar to Case I above applies.

*Case III :*  $(N_0, \mathfrak{n})$  is reduced and  $\dim \mathcal{Z}(\mathfrak{n}) = 1$ . In this case one of the statements of Proposition 4.4 must hold. If statement (b) of Proposition 4.4 holds, then by Proposition 4.5 there is nothing left to prove. Next suppose that statement (a) of Proposition 4.4 holds. If the restriction of  $\pi^\infty$  to  $\mathcal{Z}(\mathfrak{n})$  is trivial, then an argument similar to Case II above applies. If the restriction of  $\pi^\infty$  to  $\mathcal{Z}(\mathfrak{n})$  is not trivial, then from Proposition 5.1 it follows that there exists an irreducible unitary representation  $(\sigma, \rho^\sigma, \mathcal{K})$  of  $(N'_0, \mathfrak{n}')$  such that

$$(\pi, \rho^\pi, \mathcal{H}) = \text{Ind}_{(N'_0, \mathfrak{n}')}^{(N_0, \mathfrak{n})}(\sigma, \rho^\sigma, \mathcal{K}).$$

Here  $(N'_0, \mathfrak{n}')$  is the super Lie group identified by statement (a) of Proposition 4.4.

By induction hypothesis, there exists a polarizing system  $(M'_0, \mathfrak{m}', \Phi', C'_0, \mathfrak{c}', \lambda')$  of  $(N'_0, \mathfrak{n}')$  and an irreducible unitary representation  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  of  $(C'_0, \mathfrak{c}')$  which is consistent with  $(M'_0, \mathfrak{m}', \Phi', C'_0, \mathfrak{c}', \lambda')$  such that

$$(\sigma, \rho^\sigma, \mathcal{K}) = \text{Ind}_{(M'_0, \mathfrak{m}')}^{(N'_0, \mathfrak{n}')} (\sigma_\mu \circ \Phi', \rho^{\sigma_\mu \circ \Phi'}, \mathcal{K}_\mu).$$

By Proposition 3.1 we have

$$(6.3) \quad (\pi, \rho^\pi, \mathcal{H}) = \text{Ind}_{(M'_0, \mathfrak{m}')}^{(N_0, \mathfrak{n})} (\sigma_\mu \circ \Phi', \rho^{\sigma_\mu \circ \Phi'}, \mathcal{K}_\mu).$$

Let  $\tilde{\lambda}$  be an arbitrary  $\mathbb{R}$ -linear extension of  $\lambda'$  to  $\mathfrak{n}_0$ , and set

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda) = (M'_0, \mathfrak{m}', \Phi', C'_0, \mathfrak{c}', \tilde{\lambda}).$$

To show that  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  is a polarizing system of  $(N_0, \mathfrak{n})$ , it suffices to check that  $\mathfrak{m}_0$  is a polarizing subalgebra of  $\mathfrak{n}_0$  corresponding to  $\lambda$ . Let  $X, Y, Z \in \mathfrak{n}_0$  be chosen as in part (a) of Proposition 4.4. From  $Z \in \mathcal{Z}(\mathfrak{n}'_0)$  and the fact that  $\mathfrak{m}_0$  is a polarizing subalgebra of  $\mathfrak{n}'_0$  corresponding to  $\lambda'$ , it follows that  $Z \in \mathfrak{m}_0$ . Hence for every  $t \in \mathbb{R}$  and  $v \in \mathcal{K}_\mu$ ,

$$(\sigma_\mu \circ \Phi(\exp(tZ)))v = e^{t\lambda(Z)\sqrt{-1}}v.$$

Using (6.3) and the realization of induced representations given in Section 3.1 it is easy to check that for every  $t \in \mathbb{R}$  and  $v \in \mathcal{H}$ ,

$$\pi(\exp(tZ))v = e^{t\lambda(Z)\sqrt{-1}}v.$$

Since the restriction of  $\pi^\infty$  to  $\mathcal{Z}(\mathfrak{n})$  is assumed to be nontrivial, it follows that  $\lambda(Z) \neq 0$ .

Consider the skew-symmetric bilinear form

$$\omega_\lambda : \mathfrak{n}_0 \times \mathfrak{n}_0 \rightarrow \mathbb{R}$$

defined by  $\omega_\lambda(V, W) = \lambda([V, W])$ , and let  $\omega'_\lambda$  be the restriction of  $\omega_\lambda$  to  $\mathfrak{n}'_0 \times \mathfrak{n}'_0$ . Since  $\mathfrak{m}_0$  is a maximal isotropic subspace of  $\mathfrak{n}'_0$ , we have

$$\dim \mathfrak{m}_0 = \frac{1}{2}(\dim \mathfrak{n}'_0 + \dim \mathfrak{s}'_\lambda)$$

where  $\mathfrak{s}'_\lambda$  is the radical of  $\omega'_\lambda$ . To show that  $\mathfrak{m}_0$  is a maximal isotropic subspace of  $\omega_\lambda$ , it suffices to prove that

$$\dim \mathfrak{s}_\lambda = \dim \mathfrak{s}'_\lambda - 1$$

where  $\mathfrak{s}_\lambda$  is the radical of  $\omega_\lambda$ . Let  $V \in \mathfrak{s}_\lambda$ , and write  $V = aX + W$  where  $a \in \mathbb{R}$  and  $W \in \mathfrak{n}'_0$ . From  $[Y, \mathfrak{n}'_0] = \{0\}$  it follows that

$$\omega_\lambda(V, Y) = \lambda([V, Y]) = a\lambda(Z)$$

which implies that  $a = 0$ , i.e.  $V \in \mathfrak{n}'_0$ . Consequently,  $\mathfrak{s}_\lambda \subseteq \mathfrak{s}'_\lambda$ . Moreover,  $[Y, \mathfrak{n}'_0] = \{0\}$  implies that  $Y \in \mathfrak{s}'_\lambda$ , but  $\lambda([X, Y]) \neq 0$  implies that  $Y \notin \mathfrak{s}_\lambda$ . Thus  $\dim \mathfrak{s}_\lambda < \dim \mathfrak{s}'_\lambda$ , from which it readily follows that  $\dim \mathfrak{s}_\lambda = \dim \mathfrak{s}'_\lambda - 1$ . Finally, verifying that  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  is consistent with  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  is trivial.

Next we prove part (b) of Theorem 6.2. Suppose that  $\chi : C_0 \rightarrow \mathbb{C}^\times$  (respectively,  $\chi' : C'_0 \rightarrow \mathbb{C}^\times$ ) is the central character of  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  (respectively,  $(\sigma_{\mu'}, \rho^{\sigma_{\mu'}}, \mathcal{K}_{\mu'})$ ). Since  $\mathfrak{m}_0$  is a polarizing subalgebra of  $\mathfrak{n}_0$  corresponding to  $\lambda$  and  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  is consistent with  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$ , the representation  $\text{Ind}_{M_0}^{N_0} \chi \circ \Phi$  is irreducible. Since  $\pi = \text{Ind}_{M_0}^{N_0} \sigma_\mu \circ \Phi$ , it follows that the unitary representation  $(\pi, \mathcal{H})$  of the nilpotent Lie group  $N_0$  is a direct sum of  $\dim \mathcal{K}_\mu$  copies of  $\text{Ind}_{M_0}^{N_0} \chi \circ \Phi$ . With a similar argument, one can see that  $(\pi, \mathcal{H})$  is a direct sum of  $\dim \mathcal{K}_{\mu'}$  copies of the irreducible unitary representation  $\text{Ind}_{M'_0}^{N_0} \chi' \circ \Phi'$ . Consequently,  $\dim \mathcal{K}_\mu = \dim \mathcal{K}_{\mu'}$ , which immediately implies that  $(C_0, \mathfrak{c})$  and  $(C'_0, \mathfrak{c}')$  are isomorphic. Moreover, we have

$$\text{Ind}_{M_0}^{N_0} \chi \circ \Phi \simeq \text{Ind}_{M'_0}^{N_0} \chi' \circ \Phi'$$

and Kirillov theory for nilpotent Lie groups (e.g., [CG, Theorem 2.2.4]) implies that

$$\lambda' = \text{Ad}^*(n)(\lambda)$$

for some  $n \in N_0$ . □

**Corollary 6.3.** *Let  $(\pi, \rho^\pi, \mathcal{H})$  be an irreducible unitary representation of a nilpotent Lie supergroup  $(N_0, \mathfrak{n})$ , and  $(\pi, \mathcal{H})$  be the unitary representation of  $N_0$  obtained as restriction of  $(\pi, \rho^\pi, \mathcal{H})$  to the even part. Then there exists an irreducible unitary representation  $(\sigma, \mathcal{K})$  of  $N_0$  such that  $(\pi, \mathcal{H})$  is a direct sum of  $2^l$  copies of  $(\sigma, \mathcal{K})$ , where  $l$  is a nonnegative integer.*

*Proof.* Since special induction commutes with restriction to the even part, this follows immediately from part (a) of Theorem 6.2 and the fact that  $\dim \mathcal{K}_\mu = 2^l$  for some  $l \geq 0$ . □

**Remark.** Suppose that an irreducible unitary representation  $(\pi, \rho^\pi, \mathcal{H})$  is given by (6.2). If we set  $\kappa(\pi, \rho^\pi, \mathcal{H}) = \dim \mathfrak{c}$ , then by part (b) of Theorem 6.2 the positive integer  $\kappa(\pi, \rho^\pi, \mathcal{H})$  does not depend on the choice of the polarizing system and hence is an invariant of  $(\pi, \rho^\pi, \mathcal{H})$ . In fact using Corollary 6.2 one can see that  $\kappa(\pi, \rho^\pi, \mathcal{H})$  can be obtained as follows. Consider the representation  $(\pi, \mathcal{H})$  of the Lie group  $N_0$  obtained by restriction of  $(\pi, \rho^\pi, \mathcal{H})$  to the even part of  $(N_0, \mathfrak{n})$ . The representation  $(\pi, \mathcal{H})$  is always a direct sum of  $2^r$  copies of an irreducible unitary representation  $(\pi', \mathcal{H}')$  of  $N_0$ , where  $r$  is a nonnegative integer. In the latter case, we have

$$\kappa(\pi, \rho^\pi, \mathcal{H}) = \begin{cases} 2r & \text{if } (\pi, \rho^\pi, \mathcal{H}) \simeq (\pi, \rho^\pi, \Pi \mathcal{H}), \\ 2r + 1 & \text{otherwise.} \end{cases}$$

In particular, when  $r = 0$  the representation  $(\pi, \rho^\pi, \mathcal{H})$  is purely even and therefore  $\kappa(\pi, \rho^\pi, \mathcal{H}) = 1$ .

**6.2. Irreducibility of codimension-one induction.** In this section we prove that induction from a polarizing system always yields an irreducible unitary representation.

**Theorem 6.4.** *Let  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  be a polarizing system of  $(N_0, \mathfrak{n})$ . Suppose that  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  is the representation of  $(C_0, \mathfrak{c})$  consistent with this polarizing system. Then the unitary representation*

$$(\pi, \rho^\pi, \mathcal{H}) = \text{Ind}_{(M_0, \mathfrak{m})}^{(N_0, \mathfrak{n})}(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$$

*is irreducible.*

*Proof.* We prove the theorem by induction on  $\dim \mathfrak{n}$ . If  $\lambda = 0$ , then  $\mathfrak{m} = \mathfrak{n}$  and  $\ker \Phi \supseteq \mathfrak{n}_0$ , which implies that  $\mathfrak{c} = \{0\}$  and therefore  $(\pi, \rho^\pi, \mathcal{H})$  is the trivial representation. Without loss of generality, from now on we assume that  $\lambda \neq 0$ . There are three cases to consider.

*Case I :  $(N_0, \mathfrak{n})$  is not reduced.* Recall that  $\mathfrak{a}[\mathfrak{n}]$  is a  $\mathbb{Z}_2$ -graded ideal of  $\mathfrak{n}$ . Since  $\mathfrak{a}[\mathfrak{n}] \neq \{0\}$ , we have  $\mathcal{Z}(\mathfrak{n}) \cap \mathfrak{a}[\mathfrak{n}] \neq \{0\}$ . Indeed let  $\mathfrak{e}^{(0)} = \mathfrak{a}[\mathfrak{n}]$  and for any positive integer  $j$ , set  $\mathfrak{e}^{(j+1)} = [\mathfrak{n}, \mathfrak{e}^{(j)}]$ . Let  $j_0 = \min\{j \mid \mathfrak{e}^{(j)} = \{0\}\}$ . Then  $\mathfrak{e}^{(j_0-1)} \subseteq \mathfrak{a}[\mathfrak{n}] \cap \mathcal{Z}(\mathfrak{n})$ .

Let  $W \in \mathcal{Z}(\mathfrak{n}) \cap \mathfrak{a}[\mathfrak{n}]$ . Since  $\mathcal{Z}(\mathfrak{n}) \cap \mathfrak{a}[\mathfrak{n}]$  is  $\mathbb{Z}_2$ -graded, we can choose  $W$  suitably such that  $W \in \mathfrak{n}_0$  or  $W \in \mathfrak{n}_1$ . If  $W \in \mathfrak{n}_1$  then obviously  $W \in \mathfrak{m}_1$ , and if  $W \in \mathfrak{n}_0$ , then  $W \in \mathfrak{m}_0$  because otherwise  $\mathfrak{m}'_0 = \mathfrak{m}_0 + \mathbb{R}W$  is a Lie subalgebra of  $\mathfrak{n}_0$  with the property that  $\lambda([\mathfrak{m}'_0, \mathfrak{m}'_0]) = \{0\}$ , and the latter implies that  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  does not satisfy part (b) of Definition 6.1. Therefore we have shown that  $\mathcal{Z}(\mathfrak{n}) \cap \mathfrak{a}[\mathfrak{n}] \subseteq \mathfrak{m}$ .

Our next task is to show that  $\Phi(W) = 0$  for every  $W \in \mathcal{Z}(\mathfrak{n}) \cap \mathfrak{a}[\mathfrak{n}]$ . Without loss of generality, we can assume  $W \in \mathfrak{n}_0$  or  $W \in \mathfrak{n}_1$ . If  $W \in \mathfrak{n}_1$ , then we have  $[W, W] = 0$  which implies that  $[\Phi(W), \Phi(W)] = 0$ . Since  $(C_0, \mathfrak{c})$  is reduced, we have  $\Phi(W) = 0$ . If  $W \in \mathfrak{n}_0$  then for every  $v \in \mathcal{K}_\mu$  we have

$$(\sigma_\mu^\infty \circ \Phi(W))v = \mu \circ \Phi(W)\sqrt{-1}v.$$

Since  $\lambda \neq 0$ , by Lemma 2.4 and surjectivity of  $\Phi$  it follows that  $\mu \neq 0$ . From the realization of induced representations given in Section 3.1 and the fact that  $W \in \mathcal{Z}(\mathfrak{n})$  it is easily seen that for every  $v \in \mathcal{H}$ ,

$$\pi^\infty(W)v = \lambda(W)\sqrt{-1}v = \mu \circ \Phi(W)\sqrt{-1}v.$$

If  $\Phi(W) \neq 0$ , then  $\mu \circ \Phi(W) \neq 0$  from which it follows that  $\pi^\infty(W) \neq 0$ , which contradicts the fact that by Proposition 4.2 we have  $\pi^\infty(W) = 0$ .

Set  $\mathfrak{s} = \mathcal{Z}(\mathfrak{n}) \cap \mathfrak{a}[\mathfrak{n}]$  and consider the super Lie group  $(N'_0, \mathfrak{n}')$  where  $\mathfrak{n}' = \mathfrak{n}/\mathfrak{s}$ . (Thus  $N'_0 = N_0/S_0$  where  $S_0 = \{\exp(tV) \mid V \in \mathfrak{s}_0\}$ .) Obviously  $(\pi, \rho^\pi, \mathcal{H})$  factors through  $(N'_0, \mathfrak{n}')$ . We denote this representation of  $(N'_0, \mathfrak{n}')$  by  $(\bar{\pi}, \bar{\rho}^\pi, \mathcal{H})$ . Moreover,

$$\mathcal{Z}(\mathfrak{n}) \cap \mathfrak{a}[\mathfrak{n}] \subseteq \mathcal{Z}(\mathfrak{n}) \subseteq \mathfrak{m}.$$

Since  $\ker \Phi \cap \mathfrak{m}_0 = \ker \lambda \cap \mathfrak{m}_0$  we have  $\mathcal{Z}(\mathfrak{n}) \cap \mathfrak{a}[\mathfrak{n}] \cap \mathfrak{n}_0 \subseteq \ker \lambda$ . Therefore the polarizing system  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  corresponds via the quotient map

$\mathbf{q} : \mathbf{n} \rightarrow \mathbf{n}'$  to a polarizing system  $(M'_0, \mathbf{m}', \Phi', C_0, \mathbf{c}, \lambda')$  of  $(N'_0, \mathbf{n}')$ . Moreover,  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  is consistent with  $(M'_0, \mathbf{m}', \Phi', C_0, \mathbf{c}, \lambda')$ . We can express  $(\bar{\pi}, \rho^{\bar{\pi}}, \mathcal{H})$  as

$$(\bar{\pi}, \rho^{\bar{\pi}}, \mathcal{H}) = \text{Ind}_{(M'_0, \mathbf{m}')}^{(N'_0, \mathbf{n}')} (\sigma_\mu \circ \Phi', \rho^{\sigma_\mu} \circ \Phi', \mathcal{K}_\mu).$$

Since  $\dim \mathbf{n}' < \dim \mathbf{n}$ , by the induction hypothesis it follows that  $(\bar{\pi}, \rho^{\bar{\pi}}, \mathcal{H})$  (and hence  $(\pi, \rho^\pi, \mathcal{H})$ ) is irreducible.

*Case II :*  $(N_0, \mathbf{n})$  is reduced and  $\mathcal{Z}(\mathbf{n}) \cap \ker \lambda \neq \{0\}$ . In this case  $\mathcal{Z}(\mathbf{n}) \cap \ker \lambda$  is an ideal of  $\mathbf{n}$ , and the fact that  $\mathbf{m}_0$  is a polarizing subalgebra of  $\mathbf{n}_0$  corresponding to  $\lambda$  implies that  $\mathcal{Z}(\mathbf{n}) \subseteq \mathbf{m}_0$ . The representation  $(\pi, \rho^\pi, \mathcal{H})$  factors through  $(N'_0, \mathbf{n}')$  where

$$\mathbf{n}' = \mathbf{n} / \mathcal{Z}(\mathbf{n}) \cap \ker \lambda,$$

and the polarizing system  $(M_0, \mathbf{m}, \Phi, C_0, \mathbf{c}, \lambda)$  corresponds via the quotient map  $\mathbf{q} : \mathbf{n} \rightarrow \mathbf{n}'$  to a polarizing system  $(M'_0, \mathbf{m}', \Phi', C_0, \mathbf{c}, \lambda')$  of  $(N'_0, \mathbf{n}')$ . The rest of the argument is similar to Case I.

*Case III :*  $(N_0, \mathbf{n})$  is reduced and  $\mathcal{Z}(\mathbf{n}) \cap \ker \lambda = \{0\}$ . It follows that  $\dim \mathcal{Z}(\mathbf{n}) = 1$ , hence one of the statements of Proposition 4.4 should hold. If statement (b) of Proposition 4.4 holds, then there is essentially nothing left to prove. From now on we assume that statement (a) of Proposition 4.4 holds. Let  $X, Y, Z, \mathbf{n}'$ , and  $\mathbf{w}$  be as in part (a) of Proposition 4.4.

Our first task is to show that without loss of generality, we can assume that  $\lambda(Y) = 0$  and  $\lambda(Z) \neq 0$ . Indeed one can modify the choice of the polarizing system as follows. Since  $\mathbf{m}_0$  is a polarizing Lie subalgebra of  $\mathbf{n}_0$  corresponding to  $\lambda$ , we should have  $Z \in \mathbf{m}_0$ , and since  $\mathcal{Z}(\mathbf{n}) \cap \ker \lambda = \{0\}$ , we should have  $\lambda(Z) \neq 0$ . For every  $n \in N_0$ , we have a polarizing system

$$(nM_0n^{-1}, \text{Ad}(n)(\mathbf{m}), \Phi \circ \text{Ad}(n^{-1}), C_0, \mathbf{c}, \text{Ad}^*(n)(\lambda))$$

in  $(N_0, \mathbf{n})$  and one can see that

$$(\pi, \rho^\pi, \mathcal{H}) \simeq \text{Ind}_{(nM_0n^{-1}, \text{Ad}(n)(\mathbf{m}))}^{(N_0, \mathbf{n})} (\sigma_\mu \circ \Phi \circ \text{Ad}(n^{-1}), \rho^{\sigma_\mu \circ \Phi \circ \text{Ad}(n^{-1})}, \mathcal{K}_\mu).$$

In particular, if we set  $n = \exp(t_\circ X)$  where  $t_\circ = \frac{\lambda(Y)}{\lambda(Z)}$ , then

$$(\text{Ad}^*(n)(\lambda))(Y) = \lambda(Y - \frac{\lambda(Y)}{\lambda(Z)}Z) = 0.$$

The condition  $\mathcal{Z}(\mathbf{n}) \cap \ker (\text{Ad}^*(n)(\lambda)) = \{0\}$  is easy to check as well.

From now on we assume that  $\lambda(Y) = 0$  and  $\lambda(Z) \neq 0$ . Our next task is to prove that without loss of generality we can also assume that  $\mathbf{m} \subseteq \mathbf{n}'$ . Suppose, on the contrary, that  $\mathbf{m} \not\subseteq \mathbf{n}'$ . In this case we show that  $(\pi, \rho^\pi, \mathcal{H})$  is unitarily equivalent to a representation

$$(\pi', \rho^{\pi'}, \mathcal{H}') = \text{Ind}_{(M'_0, \mathbf{m}')}^{(N_0, \mathbf{n})} (\sigma_\mu \circ \Phi', \rho^{\sigma_\mu \circ \Phi'}, \mathcal{K}_\mu)$$

where  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  is consistent with a polarizing system  $(M'_0, \mathbf{m}', \Phi', C_0, \mathbf{c}, \lambda)$  which satisfies  $\mathbf{m}' \subseteq \mathbf{n}'$ . To this end, first note that in part (a) of Proposition

4.4, we can choose  $X$  such that  $\lambda(X) = 0$  and

$$\mathfrak{m} = \mathbb{R}X \oplus \mathbb{R}Z \oplus \mathfrak{w}'_0 \oplus \mathfrak{n}_1$$

where  $\mathfrak{w}'_0$  is a subspace of  $\mathfrak{n}'_0$  such that  $\lambda(\mathfrak{w}'_0) = 0$ . Indeed since  $\mathfrak{m} \not\subseteq \mathfrak{n}'$ , we can choose  $X$  such that  $X \in \mathfrak{m}_0$ . If  $\lambda(X) \neq 0$ , then since  $Z \in \mathfrak{m}$  and  $\lambda(Z) \neq 0$  we can substitute  $X$  by  $X - \frac{\lambda(X)}{\lambda(Z)}Z$ . In a similar fashion we can choose a complement  $\mathfrak{w}'_0$  to  $\mathbb{R}Z$  in  $\mathfrak{m} \cap \mathfrak{n}'$  which is included in  $\ker \lambda$ . Next note that  $Y \notin \mathfrak{m}_0$  because otherwise  $\lambda([X, Y]) = \lambda(Z) \neq 0$  which contradicts the fact that  $\mathfrak{m}_0$  is a polarizing subalgebra of  $\mathfrak{n}_0$  corresponding to  $\lambda$ . Consider the subalgebra  $\mathfrak{m}'$  of  $\mathfrak{n}'$  defined by

$$\mathfrak{m}' = \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathfrak{w}'_0 \oplus \mathfrak{n}_1.$$

To show that  $\mathfrak{m}'$  is a subalgebra of  $\mathfrak{n}'$ , note that

$$(6.4) \quad [\mathfrak{m}'_0, \mathfrak{m}'_0] \subseteq [\mathfrak{w}'_0, \mathfrak{w}'_0] \subseteq \mathfrak{m}_0 \cap \mathfrak{n}'_0 \subsetneq \mathfrak{m}'_0$$

and

$$[\mathfrak{n}_1, \mathfrak{n}_1] \subseteq \mathfrak{n}'_0 \cap \mathfrak{m}_0 \subsetneq \mathfrak{m}'_0.$$

Let  $M'_0$  be the Lie subgroup of  $N_0$  corresponding to  $\mathfrak{m}'_0$ . We define

$$\Phi' : (M'_0, \mathfrak{m}') \rightarrow (C_0, \mathfrak{c})$$

as follows. For every  $W \in \mathbb{R}Z \oplus \mathfrak{w}'_0 \oplus \mathfrak{n}_1$  and  $W' \in \mathbb{R}Y$  we set

$$\Phi'(W + W') = \Phi(W).$$

We now prove that  $(M'_0, \mathfrak{m}', \Phi', C_0, \mathfrak{c}, \lambda)$  is a polarizing system and  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  is consistent with it. From a calculation similar to (6.4) it follows that

$$\lambda([\mathfrak{m}'_0, \mathfrak{m}'_0]) \subseteq \lambda([\mathfrak{m}_0, \mathfrak{m}_0]) = \{0\}.$$

Moreover,  $Y \notin \mathfrak{m}_0$  because otherwise we have  $Z \in [\mathfrak{m}_0, \mathfrak{m}_0]$  and  $\lambda(Z) \neq 0$  which contradicts the fact that  $\mathfrak{m}_0$  is a polarizing subalgebra of  $\mathfrak{n}_0$  corresponding to  $\lambda$ . Therefore we have  $\dim \mathfrak{m}'_0 = \dim \mathfrak{m}_0$ , which implies that  $\mathfrak{m}'_0$  is a polarizing subalgebra of  $\mathfrak{n}_0$ . Using  $[Y, \mathfrak{n}'] = \{0\}$  it is easy to check that part (c) of Definition 6.1 holds. Part (d) of Definition 6.1 follows from  $\lambda(X) = \lambda(Y) = 0$  and  $\lambda(Z) \neq 0$ . Finally, one can check that  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  is consistent with  $(M'_0, \mathfrak{m}', \Phi', C_0, \mathfrak{c}, \lambda)$ .

To prove that  $(\pi, \rho^\pi, \mathcal{H}) \simeq (\pi', \rho^{\pi'}, \mathcal{H}')$ , it suffices to show that

$$(6.5) \quad \text{Ind}_{(M_0, \mathfrak{m})}^{(M''_0, \mathfrak{m}'')}(\sigma_\mu \circ \Phi, \rho^{\sigma_\mu \circ \Phi}, \mathcal{K}_\mu) \simeq \text{Ind}_{(M'_0, \mathfrak{m}')}^{(M''_0, \mathfrak{m}'')}(\sigma_\mu \circ \Phi', \rho^{\sigma_\mu \circ \Phi'}, \mathcal{K}_\mu)$$

where  $M''_0 = M_0 M'_0$  and  $\mathfrak{m}'' = \mathfrak{m} + \mathfrak{m}'$ , i.e.,

$$\mathfrak{m}'' = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathfrak{w}'_0 \oplus \mathfrak{n}_1.$$

Since  $\mathfrak{m}''$  is  $\mathbb{Z}_2$ -graded, we can express it as  $\mathfrak{m}'' = \mathfrak{m}''_0 \oplus \mathfrak{m}''_1$ . Observe that the vector space  $\mathfrak{w}'_0$  is in fact an ideal of  $\mathfrak{m}'_0$ . To prove the latter statement, note that since  $\mathfrak{m}_0$  and  $\mathfrak{m}'_0$  are polarizing subalgebras of  $\mathfrak{n}_0$  corresponding to  $\lambda$ , we should have

$$[\mathfrak{m}_0, \mathfrak{m}_0] \subseteq \mathbb{R}X \oplus \mathfrak{w}'_0 \quad \text{and} \quad [\mathfrak{m}'_0, \mathfrak{m}'_0] \subseteq \mathbb{R}Y \oplus \mathfrak{w}'_0$$

which imply that  $\mathbb{R}X \oplus \mathfrak{w}'_0$  and  $\mathbb{R}Y \oplus \mathfrak{w}'_0$  are Lie subalgebras of  $\mathfrak{n}_0$ . Since

$$\mathfrak{w}'_0 = (\mathbb{R}X \oplus \mathfrak{w}'_0) \cap (\mathbb{R}Y \oplus \mathfrak{w}'_0),$$

the vector space  $\mathfrak{w}'_0$  is in fact a Lie subalgebra of both of  $\mathbb{R}X \oplus \mathfrak{w}'_0$  and  $\mathbb{R}Y \oplus \mathfrak{w}'_0$ . But in a nilpotent Lie algebra, any Lie subalgebra of codimension one is an ideal. Therefore  $\mathfrak{w}'_0$  is an ideal in both  $\mathbb{R}X \oplus \mathfrak{w}'_0$  and  $\mathbb{R}Y \oplus \mathfrak{w}'_0$ . It follows that  $\mathfrak{w}'_0$  is an ideal in the Lie algebra generated by  $\mathbb{R}X \oplus \mathfrak{w}'_0$  and  $\mathbb{R}Y \oplus \mathfrak{w}'_0$ , i.e., in  $\mathfrak{m}''_0 = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathfrak{w}'_0$ .

Next we obtain the unitary equivalence of (6.5). Let  $E_0 = \{ \exp(tZ) \mid t \in \mathbb{R} \}$  and  $\chi : E_0 \rightarrow \mathbb{C}^\times$  be the unitary character given by

$$\chi(\exp(tZ)) = e^{t\lambda(Z)\sqrt{-1}}.$$

If  $(\pi_L, \rho^{\pi_L}, \mathcal{H}_L)$  denotes the representation on the left hand side of (6.5), then we can realize  $\mathcal{H}_L$  as  $L^2(\mathbb{R}, \mathcal{K}_\mu)$  such that the action of  $(M''_0, \mathfrak{m}'')$  is given as follows. For every  $y, t \in \mathbb{R}$ ,  $W' \in \mathfrak{w}'_0$ , and  $f \in L^2(\mathbb{R}, \mathcal{K}_\mu)$ , we have

$$\begin{aligned} (\pi_L(\exp(tX))f)(y) &= \chi(ty)f(y) \\ (\pi_L(\exp(tY))f)(y) &= f(y+t) \\ (\pi_L(\exp(tZ))f)(y) &= \chi(t)f(y) \\ (\pi_L(\exp(W'))f)(y) &= f(y). \end{aligned}$$

Moreover, if  $f \in L^2(\mathbb{R}, \mathcal{K}_\mu)$  is in the Schwartz space then from  $[Y, \mathfrak{n}_1] = \{0\}$  it follows that for every  $W \in \mathfrak{n}_1$  and  $y \in \mathbb{R}$  we have

$$(\rho^{\pi_L}(W))(y) = \Phi(\exp(yY) \cdot W)(f(y)) = \Phi(W)(f(y)).$$

Similarly, if  $(\pi_R, \rho^{\pi_R}, \mathcal{H}_R)$  denotes the representation on the right hand side of (6.5), then  $(\pi_R, \rho^{\pi_R}, \mathcal{H}_R)$  can also be realized on  $L^2(\mathbb{R}, \mathcal{K}_\mu)$  as follows. For every  $x, t \in \mathbb{R}$ ,  $W' \in \mathfrak{w}'_0$ , and  $f \in L^2(\mathbb{R}, \mathcal{K}_\mu)$ , we have

$$\begin{aligned} (\pi_R(\exp(tX))f)(x) &= f(x+t) \\ (\pi_R(\exp(tY))f)(x) &= \chi(-tx)f(x) \\ (\pi_R(\exp(tZ))f)(x) &= \chi(t)f(x) \\ (\pi_R(\exp(W'))f)(x) &= f(x). \end{aligned}$$

Moreover, if  $f \in L^2(\mathbb{R}, \mathcal{K}_\mu)$  is indeed in the Schwartz space, then for every  $W \in \mathfrak{n}_1$  and  $x \in \mathbb{R}$  we have

$$(\rho^{\pi_R}(W))(x) = \Phi'(\exp(xX) \cdot W)(f(x)) = \Phi(W)(f(x))$$

where the last equality follows from the fact that  $\Phi(X) = 0$  and thus

$$\begin{aligned} \Phi'(\exp(xX) \cdot W) &= \Phi'(W + [X, W] + \frac{1}{2}[X, [X, W]] + \cdots) \\ &= \Phi(W + [X, W] + \frac{1}{2}[X, [X, W]] + \cdots) \\ &= \Phi(W) + [\Phi(X), \Phi(W)] + \frac{1}{2}[\Phi(X), [\Phi(X), \Phi(W)]] + \cdots \\ &= \Phi(W). \end{aligned}$$

It is now easy to check that the isometry  $T : \mathcal{H}_L \rightarrow \mathcal{H}_R$  which intertwines  $(\pi_L, \rho^{\pi_L}, \mathcal{H}_L)$  and  $(\pi_R, \rho^{\pi_R}, \mathcal{H}_R)$  is given by the Fourier transform, i.e.,

$$Tf(x) = \int_{-\infty}^{\infty} \chi(xy) f(y) dy.$$

We now complete the proof of Case III. The proof closely follows an argument that is given in [CG, p. 63]. Recall that as shown above, we can assume that  $\mathfrak{m} \subseteq \mathfrak{n}'$ . It follows that  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  is a polarizing system in  $(N'_0, \mathfrak{n}')$ . Since  $\dim \mathfrak{n}' < \dim \mathfrak{n}$ , by induction hypothesis the representation

$$(6.6) \quad (\pi'', \rho^{\pi''}, \mathcal{H}'') = \text{Ind}_{(M_0, \mathfrak{m})}^{(N'_0, \mathfrak{n}')} (\sigma_\mu \circ \Phi, \rho^{\sigma_\mu \circ \Phi}, \mathcal{K}_\mu)$$

is irreducible. Since  $Z \in \mathcal{Z}(\mathfrak{n}')$ , by [CCTV, Lemma 5] there exists a real number  $b \in \mathbb{R}$  such that for every  $t \in \mathbb{R}$  and  $v \in \mathcal{H}''$  we have

$$\pi''(\exp(tZ))v = e^{tb\sqrt{-1}}v.$$

Recall that  $\lambda(Z) \neq 0$  and  $\lambda(Y) = 0$ . Since  $Z \in \mathcal{Z}(\mathfrak{n}) \cap \mathfrak{n}'_0 \subseteq \mathfrak{m}_0$ , from (6.6) and the realization of the induced representation (see Section 3.1) it follows that

$$\pi''(\exp(tZ))v = e^{t\lambda(Z)\sqrt{-1}}v$$

and therefore  $b \neq 0$ . Next observe that

$$(\pi, \rho^\pi, \mathcal{H}) = \text{Ind}_{(N'_0, \mathfrak{n}')}^{(N_0, \mathfrak{n})} (\pi'', \rho^{\pi''}, \mathcal{H}'')$$

and by Section 3.1 we can assume  $\mathcal{H} = L^2(\mathbb{R}, \mathcal{H}'')$  where for every  $f \in L^2(\mathbb{R}, \mathcal{H}'')$ ,  $s \in \mathbb{R}$ ,  $t \in \mathbb{R}$ , and  $n \in N'_0$  we have

$$(6.7) \quad (\pi(\exp(tX))f)(s) = f(s+t)$$

and

$$(\pi(n)f)(s) = \pi''(\exp(sX)n\exp(-sX))(f(s)).$$

In particular, since  $\text{Span}_{\mathbb{R}}\{X, Y, Z\}$  is a Heisenberg Lie algebra, we have

$$(\pi(\exp(tY))f)(s) = e^{stb\sqrt{-1}}f(s).$$

Moreover, if  $f \in L^2(\mathbb{R}, \mathcal{H}'')$  is a smooth vector for the action of  $\pi$  and has compact support, then for every  $W \in \mathfrak{n}_1$  and  $s \in \mathbb{R}$  we have

$$(\rho^\pi(W)f)(s) = \rho^{\pi''}(\exp(sX) \cdot W)(f(s)).$$

Let  $T : L^2(\mathbb{R}, \mathcal{H}'') \rightarrow L^2(\mathbb{R}, \mathcal{H}'')$  be a bounded even linear operator which intertwines  $(\pi, \rho^\pi, \mathcal{H})$  with itself. To complete the proof of Case III, it suffices to show that  $T$  is a scalar multiple of the identity. From [CG, Lemma 2.3.3], [CG, Lemma 2.3.2] and [CG, Lemma 2.3.1] it follows that there exists a family  $\{T_t\}_{t \in \mathbb{R}}$  of even linear operators  $T_t : \mathcal{H}'' \rightarrow \mathcal{H}''$  such that  $\|T_t\| \leq \|T\|$  for every  $t \in \mathbb{R}$ , and for every  $f \in L^2(\mathbb{R}, \mathcal{H}'')$  we have  $Tf(t) = T_t(f(t))$ . One can check that  $T_t$  intertwines the action of the representation  $(\pi_t'', \rho^{\pi_t''}, \mathcal{H}'')$  of  $(N'_0, \mathfrak{n}')$  which is defined by

$$\pi_t''(n) = \pi''(\exp(tX)n\exp(-tX)) \quad \text{and} \quad \rho^{\pi_t''}(W) = \rho^{\pi''}(\exp(tX) \cdot W).$$

But  $(\pi_t'', \rho^{\pi_t''}, \mathcal{H}'')$  is irreducible, and from [CG, Lemma 5] it follows that for every  $t \in \mathbb{R}$ , the operator  $T_t$  is multiplication by a scalar  $\gamma(t)$ . From (6.7) it follows that  $\gamma(t)$  does not depend on  $t$ , i.e.,  $T$  is a scalar multiple of identity.  $\square$

**6.3. Existence of suitable polarizing subalgebras.** In this section  $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1$  will be a nilpotent Lie superalgebra. In this technical section we prove the existence of a special kind of polarizing subalgebras in  $\mathfrak{n}_0$ . The main goal of our fairly complicated arguments is to prove Lemma 6.10.

For every  $\lambda \in \mathfrak{n}_0^*$  we consider the symmetric bilinear form

$$B_\lambda : \mathfrak{n}_1 \times \mathfrak{n}_1 \rightarrow \mathbb{R}$$

defined by  $B_\lambda(X, Y) = \lambda([X, Y])$ . We denote the radical of  $B_\lambda$  by  $\mathfrak{r}_\lambda$ .

**Lemma 6.5.** *Suppose  $\lambda \in \mathfrak{n}_0^*$  and  $B_\lambda$  is nonnegative definite. If  $X \in \mathfrak{n}_1$  is an isotropic vector, i.e., it satisfies  $B_\lambda(X, X) = 0$ , then  $X \in \mathfrak{r}_\lambda$ .*

*Proof.* Suppose, on the contrary, that there exists an element  $Y \in \mathfrak{n}_1$  such that  $\lambda([X, Y]) \neq 0$ . Then for every  $s \in \mathbb{R}$  we have

$$\lambda([X + sY, X + sY]) = \lambda([X, X]) + 2s\lambda([X, Y]) + s^2\lambda([Y, Y]).$$

Since  $\lambda([X, X]) = 0$ , one can find an  $s \in \mathbb{R}$  such that  $\lambda([X + sY, X + sY]) < 0$ , which contradicts the fact that  $B_\lambda$  is nonnegative definite.  $\square$

In the rest of this section we fix  $\lambda \in \mathfrak{n}_0^*$  such that  $B_\lambda$  is nonnegative definite. Suppose that  $\mathfrak{n}$  has a subalgebra  $\mathfrak{n}' = \mathfrak{n}'_0 \oplus \mathfrak{n}'_1$  where  $\mathfrak{n}'_0 = \mathfrak{n}_0$  and  $\dim \mathfrak{n}'_1 = \dim \mathfrak{n}_1 - 1$ . Then it is easily checked that  $\mathfrak{n}'$  is indeed an ideal of  $\mathfrak{n}$  and  $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}'$ . As a vector space, we can write  $\mathfrak{n}_1$  as a direct sum

$$(6.8) \quad \mathfrak{n}_1 = \mathfrak{n}'_1 \oplus \mathbb{R}A$$

for some  $A \in \mathfrak{n}_1$ , and without loss of generality we can assume that  $A$  is chosen suitably such that

$$(6.9) \quad B_\lambda(A, \mathfrak{n}'_1) = 0.$$

In the rest of this section we fix such an  $A \in \mathfrak{n}_1$ .

**Lemma 6.6.** *Let  $E, F \in \mathfrak{n}_1$ . Suppose that*

- (a)  $B_\lambda([A, E], [A, E], [F], [A, E], [F]) = 0$ ,
- (b) *For every  $G \in \mathfrak{n}_1$ , we have  $B_\lambda(X_G, X_G) = 0$  where*

$$X_G = [[A, [A, [F, [F, [A, E]]]]], G].$$

*Then  $B_\lambda([A, E], F, [A, E], F) = 0$ .*

*Proof.* Set  $Y = [A, E]$ . Our goal is to prove that

$$B_\lambda([F, Y], [F, Y]) = 0.$$

Observe that by the Jacobi identity we have

$$(6.10) \quad \lambda([F, Y], [F, Y]) - \lambda([F, [Y, [F, Y]]) + \lambda([Y, [[F, Y], F]]) = 0.$$

Set  $P = [[F, Y], F]$ . Then

$$\lambda([Y, [[F, Y], F]]) = \lambda([A, E], P) = -\lambda([P, [A, E]])$$

and by the Jacobi identity we have

$$(6.11) \quad \lambda([P, [A, E]]) + \lambda([A, [E, P]]) - \lambda([E, [P, A]]) = 0.$$

Since  $[E, P] \in \mathfrak{n}'_1$ , by (6.9) we have  $\lambda([A, [E, P]]) = 0$ . To complete the proof of the lemma it suffices to prove that  $\lambda([F, [Y, [F, Y]]) = 0$  and  $\lambda([E, [P, A]]) = 0$ . By Lemma 6.5 it suffices to show that

$$[Y, [Y, F]] \quad \text{and} \quad [A, P] = [A, [F, [F, Y]]]$$

are isotropic vectors for  $B_\lambda$ . For  $[Y, [Y, F]]$ , the latter statement is assumption (a) of the lemma. Next we prove that

$$B_\lambda([A, P], [A, P]) = 0.$$

By the Jacobi identity we have

$$(6.12) \quad \lambda([A, P], [A, P]) - \lambda([A, [P, [A, P]]]) + \lambda([P, [[A, P], A]]) = 0.$$

Since  $[P, [A, P]] \in \mathfrak{n}'_1$ , by (6.9) we have

$$(6.13) \quad \lambda([A, [P, [A, P]]]) = 0.$$

By the Jacobi identity we have

$$(6.14) \quad \begin{aligned} & \lambda([A, [A, P]], [F, [F, Y]]) + \lambda([F, [[F, Y], [A, [A, P]]]]) \\ & - \lambda([F, Y], [[A, [A, P]], F]) = 0. \end{aligned}$$

Next observe that

$$\lambda([F, [[F, Y], [A, [A, P]]]]) = -\lambda([F, [[A, [A, P]], [F, Y]]])$$

and  $[A, [A, P]] = [A, [A, [F, [F, [A, E]]]]]$ . Therefore from assumption (b) of Lemma 6.6 and Lemma 6.5 it follows that

$$\lambda([F, [[A, [A, P]], [F, Y]]]) = 0.$$

A similar argument proves that

$$\lambda([F, Y], [[A, [A, P]], F]) = 0.$$

The last two equalities, together with (6.14), imply that

$$(6.15) \quad \lambda([A, [A, P]], [F, [F, Y]]) = 0.$$

But

$$[[A, [A, P]], [F, [F, Y]]] = [[A, [A, P]], P] = -[P, [A, [A, P]]]$$

and therefore from (6.12), (6.13), and (6.15) it follows that

$$B_\lambda([A, P], [A, P]) = 0$$

which completes the proof. □

**Lemma 6.7.** *Let  $E, F \in \mathfrak{n}_1$ . Then  $B_\lambda([A, E], F, [[A, E], F]) = 0$ .*

*Proof.* Set  $\mathfrak{n}^{(0)} = \mathfrak{n}$  and  $\mathfrak{n}^{(i)} = [\mathfrak{n}^{(0)}, \mathfrak{n}^{(i-1)}]$ . Note that  $\mathfrak{n}^{(r)} = \{0\}$  for  $r \gg 0$ . We prove the lemma by a backward induction as follow. We assume that the lemma holds for every  $E, F$  such that  $[[A, E], F] \in \mathfrak{n}^{(r)}$ , and we prove that it holds for every  $E, F$  such that  $[[A, E], F] \in \mathfrak{n}^{(r-1)}$ .

Assume the induction hypothesis, and consider  $E, F \in \mathfrak{n}_1$  such that

$$[[A, E], F] \in \mathfrak{n}^{(r-1)}.$$

It is easily seen that  $[[A, E], [[A, E], F]]$  and every element of the form

$$[[A, [A, [F, [F, [A, E]]]]], G]$$

where  $G \in \mathfrak{n}_1$  satisfy the induction hypothesis and therefore they are isotropic vectors for  $B_\lambda$ . Lemma 6.6 implies that  $[[A, E], F]$  is an isotropic vector for  $B_\lambda$  as well. □

**Lemma 6.8.** *Let  $E, F \in \mathfrak{n}_1$ . Then  $B_\lambda([A, [E, F]], [A, [E, F]]) = 0$ .*

*Proof.* Set  $X = [E, F]$ . By the Jacobi identity we have

$$\lambda([A, X], [A, X]) - \lambda([A, [X, [A, X]]]) + \lambda([X, [[A, X], A]]) = 0.$$

Since  $[X, [A, X]] \in \mathfrak{n}'_1$ , we have  $\lambda([A, [X, [A, X]]]) = 0$  and therefore

$$(6.16) \quad \lambda([A, X], [A, X]) = -\lambda([X, [[A, X], A]]) = -\lambda([X, [A, [A, X]]]).$$

To complete the proof of the lemma it suffices to show that the rightmost term in (6.16) vanishes. By the Jacobi identity we have

$$(6.17) \quad \begin{aligned} \lambda([A, [A, X]], [E, F]) &+ \lambda([E, [F, [A, [A, X]]]]) \\ &- \lambda([F, [[A, [A, X]], E]]) = 0. \end{aligned}$$

From Lemma 6.7 it follows that  $[F, [A, [A, X]]]$  and  $[E, [A, [A, X]]]$  are isotropic vectors for  $B_\lambda$ , and Lemma 6.5 implies that

$$(6.18) \quad \lambda([E, [F, [A, [A, X]]]]) = 0 \quad \text{and} \quad \lambda([F, [[A, [A, X]], E]]) = 0.$$

From (6.18) and (6.17) we obtain

$$\lambda([X, [A, [A, X]]]) = -\lambda([A, [A, X]], [E, F]) = 0$$

which completes the proof of the lemma. □

**Lemma 6.9.** *We have  $[\mathfrak{n}_1, \mathfrak{n}_1] \subseteq \mathfrak{r}_\lambda$ .*

*Proof.* We prove the statement by induction on the dimension of  $\mathfrak{n}$ . Since  $\mathfrak{n}$  is nilpotent, it has a subalgebra  $\mathfrak{n}' = \mathfrak{n}'_0 \oplus \mathfrak{n}'_1$  of codimension one. If  $\mathfrak{n}'_1 = \mathfrak{n}_1$ , then the lemma follows by the induction hypothesis applied to  $\mathfrak{n}'$ . If  $\dim \mathfrak{n}'_1 = \dim \mathfrak{n}_1 - 1$ , then we have shown that we can find an element  $A \in \mathfrak{n}_1$  such that we have a direct sum decomposition such as (6.8) for which (6.9) holds. One can check that

$$(6.19) \quad [[\mathfrak{n}_1, \mathfrak{n}_1], [\mathfrak{n}_1, \mathfrak{n}_1]] = [[\mathfrak{n}'_1, \mathfrak{n}'_1], [\mathfrak{n}'_1, \mathfrak{n}'_1]] + [[A, \mathfrak{n}_1], [\mathfrak{n}_1, \mathfrak{n}_1]].$$

To complete the proof of Lemma 6.9, we need to show that

$$[[\mathfrak{n}'_1, \mathfrak{n}'_1], [\mathfrak{n}'_1, \mathfrak{n}'_1]] \subseteq \ker \lambda \quad \text{and} \quad [[A, \mathfrak{n}_1], [\mathfrak{n}_1, \mathfrak{n}_1]] \subseteq \ker \lambda.$$

Induction hypothesis applied to  $\mathfrak{n}'$  immediately implies that

$$[[\mathfrak{n}'_1, \mathfrak{n}'_1], [\mathfrak{n}'_1, \mathfrak{n}'_1]] \subseteq \ker \lambda.$$

Next we show that

$$[[A, \mathfrak{n}_1], [\mathfrak{n}_1, \mathfrak{n}_1]] \subseteq \ker \lambda.$$

To this end we prove that for every  $B, C, D \in \mathfrak{n}_1$  we have

$$(6.20) \quad \lambda([[A, B], [C, D]]) = 0.$$

By the Jacobi identity we have

$$\lambda([[[C, D], [A, B]]]) + \lambda([A, [B, [C, D]]]) - \lambda([B, [[C, D], A]]) = 0.$$

But as  $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{n}'$ , we have  $[B, [C, D]] \in \mathfrak{n}'_1$  and therefore

$$\lambda([A, [B, [C, D]]]) = 0.$$

By Lemma 6.5, to prove (6.20) it suffices to show that

$$B_\lambda([A, [C, D]], [A, [C, D]]) = 0.$$

The latter statement follows from Lemma 6.8. □

**Lemma 6.10.** *There exists a polarizing subalgebra  $\mathfrak{m}_0$  of  $\mathfrak{n}_0$  corresponding to  $\lambda$  such that  $\mathfrak{m}_0 \supseteq [\mathfrak{n}_1, \mathfrak{n}_1]$ .*

*Proof.* Since  $[\mathfrak{n}_1, \mathfrak{n}_1]$  is an ideal of  $\mathfrak{n}_0$ , we can find a sequence of ideals of  $\mathfrak{n}_0$  such as

$$\mathfrak{n}_0 = \mathfrak{i}^{(1)} \supset \mathfrak{i}^{(2)} \supset \mathfrak{i}^{(3)} \supset \dots \supset \mathfrak{i}^{(r-1)} \supset \mathfrak{i}^{(r)} = \{0\}$$

such that for every  $1 < j \leq r$  we have  $\dim \mathfrak{i}^{(j-1)} = \dim \mathfrak{i}^{(j)} + 1$  and moreover for some  $1 \leq s \leq r$  we have  $[\mathfrak{n}_1, \mathfrak{n}_1] = \mathfrak{i}^{(s)}$ . For every  $1 \leq j \leq r$  let

$$\omega_\lambda^{(j)} : \mathfrak{i}^{(j)} \times \mathfrak{i}^{(j)} \rightarrow \mathbb{R}$$

be the skew-symmetric bilinear form defined by  $\omega_\lambda^{(j)}(X, Y) = \lambda([X, Y])$  and let  $\mathfrak{q}^{(j)}$  be the radical of  $\omega_\lambda^{(j)}$ .

By a result of M. Vergne (see [CG, Theorem 1.3.5]) the vector space

$$\mathfrak{q}^{(1)} + \dots + \mathfrak{q}^{(r)}$$

is indeed a polarizing Lie subalgebra of  $\mathfrak{n}_0$  corresponding to  $\lambda$ . Lemma 6.9 implies that  $\mathfrak{q}^{(s)} \supseteq [\mathfrak{n}_1, \mathfrak{n}_1]$ . □

**6.4. Existence of polarizing systems.** Throughout this section  $(N_0, \mathfrak{m})$  will be a nilpotent super Lie group. Let  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  be a polarizing system in  $(N_0, \mathfrak{n})$  and  $(\sigma_\mu, \rho_\mu^\sigma, \mathcal{K}_\mu)$  be a representation of  $(C_0, \mathfrak{c})$  which is consistent with this polarizing system. Let  $B_\lambda$  denote the bilinear form on  $\mathfrak{n}_1$  defined in Section 6.3.

Obviously, for every  $X \in \mathfrak{n}_1$  we have

$$B_\lambda(X, X) = \lambda([X, X]) = \mu \circ \Phi([X, X]) = \mu([\Phi(X), \Phi(X)]) \geq 0.$$

Consequently,  $B_\lambda$  is nonnegative definite.

Conversely, let  $\lambda \in \mathfrak{n}_0^*$  be such that  $B_\lambda$  is nonnegative definite. From Lemma 6.10 it follows that there exists a sub super Lie group  $(M_0, \mathfrak{m})$  of  $(N_0, \mathfrak{n})$  such that  $\mathfrak{m}_1 = \mathfrak{n}_1$  and  $\mathfrak{m}_0$  is a polarizing Lie subalgebra of  $\mathfrak{n}_0$  corresponding to  $\lambda$ . Let

$$\mathfrak{k}_\lambda = \{ X \in \mathfrak{m}_0 \mid \lambda(X) = 0 \}$$

and set  $\mathfrak{j} = \mathfrak{k}_\lambda \oplus \mathfrak{r}_\lambda$ .

**Lemma 6.11.** *The vector space  $\mathfrak{j}$  is an ideal in  $\mathfrak{m}$ .*

*Proof.* Since  $\lambda([\mathfrak{m}_0, \mathfrak{m}_0]) = 0$ , we have  $[\mathfrak{m}_0, \mathfrak{m}_0] \subseteq \mathfrak{k}_\lambda$  and therefore  $[\mathfrak{m}_0, \mathfrak{k}_\lambda] \subseteq \mathfrak{k}_\lambda$ .

Next we prove that  $[\mathfrak{k}_\lambda, \mathfrak{m}_1] \subseteq \mathfrak{r}_\lambda$ . To this end, first note that by the Jacobi identity for every  $A \in \mathfrak{k}_\lambda$ ,  $B \in \mathfrak{r}_\lambda$ , and  $C \in \mathfrak{n}_1$  we have

$$-[[A, B], C] - [[B, C], A] + [[C, A], B] = 0$$

and therefore

$$-\lambda([[A, B], C]) - \lambda([[B, C], A]) + \lambda([[C, A], B]) = 0.$$

But  $\lambda([[B, C], A]) = 0$  because  $[B, C] \in [\mathfrak{n}_1, \mathfrak{n}_1] \subseteq \mathfrak{m}_0$ , and

$$\lambda([[C, A], B]) = 0$$

because  $B \in \mathfrak{r}_\lambda$ . Consequently, for every  $A \in \mathfrak{k}_\lambda$  and  $B \in \mathfrak{r}_\lambda$  we have  $\text{ad}_A B \in \mathfrak{r}_\lambda$ . It follows that  $\text{ad}_A$  descends to a linear transformation  $\overline{\text{ad}}_A : \mathfrak{n}_1/\mathfrak{r}_\lambda \rightarrow \mathfrak{n}_1/\mathfrak{r}_\lambda$ . The bilinear form  $B_\lambda$  induces a positive definite bilinear form

$$\overline{B}_\lambda : \mathfrak{n}_1/\mathfrak{r}_\lambda \times \mathfrak{n}_1/\mathfrak{r}_\lambda \rightarrow \mathbb{R}.$$

Next observe that for every  $A \in \mathfrak{k}_\lambda$  and every  $V, W \in \mathfrak{n}_1$  we have

$$-\lambda([W, [A, V]]) + \lambda([A, [V, W]]) + \lambda([V, [W, A]]) = 0.$$

Moreover,  $\lambda([A, [V, W]]) = 0$  since  $[A, [V, W]] \in [\mathfrak{m}_0, \mathfrak{m}_0] \subseteq \mathfrak{k}_\lambda$ . Therefore for every  $v, w \in \mathfrak{n}_1/\mathfrak{r}_\lambda$  we have

$$\overline{B}_\lambda(\overline{\text{ad}}_A v, w) = -\overline{B}_\lambda(v, \overline{\text{ad}}_A w).$$

In other words,  $\overline{\text{ad}}_A$  is skew-symmetric. Since  $\overline{\text{ad}}_A$  is also nilpotent, it follows that  $\overline{\text{ad}}_A = 0$ . Therefore  $[\mathfrak{k}_\lambda, \mathfrak{n}_1] \subseteq \mathfrak{r}_\lambda$ .

Next we prove that  $[\mathfrak{m}_0, \mathfrak{r}_\lambda] \subseteq \mathfrak{r}_\lambda$ . To this end, first note that by the Jacobi identity, for every  $A \in \mathfrak{m}_0$ ,  $B \in \mathfrak{r}_\lambda$ , and  $C \in \mathfrak{n}_1$  we have

$$-\lambda([A, B], C) - \lambda([B, C], A) + \lambda([C, A], B) = 0.$$

But  $\lambda([B, C], A) = 0$  because  $[B, C], A \in [\mathfrak{m}_0, \mathfrak{m}_0] \subseteq \mathfrak{k}_\lambda$ , and

$$\lambda([C, A], B) = 0$$

because  $B \in \mathfrak{r}_\lambda$ . It follows that  $\lambda([A, B], C) = 0$ , and consequently, as  $C \in \mathfrak{n}_1$  is arbitrary, we have  $[A, B] \in \mathfrak{r}_\lambda$ .

Finally, the inclusion  $[\mathfrak{m}_1, \mathfrak{r}_\lambda] \subseteq \mathfrak{k}_\lambda$  follows from the definition of  $\mathfrak{k}_\lambda$ .  $\square$

**Lemma 6.12.** *The quotient Lie superalgebra  $\mathfrak{m}/\mathfrak{j}$  is reduced.*

*Proof.* It suffices to prove that for every  $X \in \mathfrak{n}_1$  such that  $[X, X] \in \mathfrak{k}_\lambda$ , we have  $X \in \mathfrak{r}_\lambda$ . But this follows immediately from Lemma 6.5.  $\square$

**Proposition 6.13.** *Let  $(M_0, \mathfrak{m})$  be a sub super Lie group of  $(N_0, \mathfrak{n})$  such that  $\mathfrak{m}_0$  is a polarizing subalgebra of  $\mathfrak{n}_0$  corresponding to  $\lambda$ . Then there exists a polarizing system  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  and a representation  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  of  $(C_0, \mathfrak{c})$  which is consistent with this polarizing system. Moreover, up to unitary equivalence and parity change the representation  $(\sigma_\mu \circ \Phi, \rho^{\sigma_\mu \circ \Phi}, \mathcal{K}_\mu)$  of  $(M_0, \mathfrak{m})$  is unique.*

*Proof.* By Lemma 2.4 the quotient Lie superalgebra  $\mathfrak{m}/\mathfrak{j}$  is either zero or has a one dimensional even part. Moreover,  $\mathfrak{m}/\mathfrak{j}$  is zero if and only if  $\lambda = 0$ . Since the case  $\lambda = 0$  can be easily dealt with, from now on we assume that  $\lambda \neq 0$ , and consequently  $\mathfrak{m}/\mathfrak{j}$  is nonzero.

Since  $\mathfrak{m}/\mathfrak{j}$  is reduced and nilpotent, we have  $\mathcal{Z}(\mathfrak{m}/\mathfrak{j}) = \mathfrak{m}_0/\mathfrak{k}_\lambda$  and hence  $\dim \mathcal{Z}(\mathfrak{m}/\mathfrak{j}) = 1$ . Therefore from Proposition 4.4 it follows that  $\mathfrak{m}/\mathfrak{j}$  is of Clifford type. (Note that one may have  $\dim \mathfrak{m}/\mathfrak{j} = 1$ .)

Let  $K_\lambda$  be a closed subgroup of  $M_0$  with Lie algebra  $\mathfrak{k}_\lambda$  and

$$\overline{\Phi} : (M_0, \mathfrak{m}) \rightarrow (M_0/K_\lambda, \mathfrak{m}/\mathfrak{j})$$

be the natural quotient map. Then  $(M_0, \mathfrak{m}, \overline{\Phi}, M_0/K_\lambda, \mathfrak{m}/\mathfrak{k}_\lambda, \lambda)$  is a polarizing system. Moreover, up to unitary equivalence and parity change, there exists a unique irreducible unitary representation  $(\sigma_{\overline{\mu}}, \rho^{\sigma_{\overline{\mu}}}, \mathcal{K}_{\overline{\mu}})$  of  $(M_0/K_\lambda, \mathfrak{m}/\mathfrak{j})$  which is consistent with this polarizing system.

Next we prove the uniqueness claim of Proposition 6.13. Without loss of generality, we can assume  $\lambda \neq 0$ . Consider another polarizing system

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$$

and a consistent irreducible unitary representation  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  of  $(C_0, \mathfrak{c})$ . Observe that

$$(6.21) \quad \mathfrak{j} \subseteq \ker \Phi.$$

Indeed for every  $X \in \mathfrak{k}_\lambda$  we have  $\mu \circ \Phi(X) = \lambda(X) = 0$  which implies that  $\Phi(X) = 0$ . Similarly, for every  $X \in \mathfrak{r}_\lambda$  we have  $\mu \circ \Phi([X, X]) = \lambda([X, X]) = 0$

which implies that  $[\Phi(X), \Phi(X)] = \Phi([X, X]) = 0$ . But since  $\mathfrak{c}$  is reduced, it follows that  $\Phi(X) = 0$ . This completes the proof of (6.21).

From (6.21) it follows that there exists an epimorphism

$$\Psi : (M_0/K_\lambda, \mathfrak{m}/\mathfrak{j}) \rightarrow (C_0, \mathfrak{c})$$

which satisfies  $\Psi \circ \overline{\Phi} = \Phi$ . However, any epimorphism between super Lie groups of Clifford type is indeed an isomorphism. From Proposition 4.5 it follows that

$$(\sigma_{\overline{\mu}} \circ \overline{\Phi}, \rho^{\sigma_{\overline{\mu}} \circ \overline{\Phi}}, \mathcal{K}_{\overline{\mu}}) \simeq (\sigma_{\mu} \circ \Psi \circ \overline{\Phi}, \rho^{\sigma_{\mu} \circ \Psi \circ \overline{\Phi}}, \mathcal{K}_{\mu}) \simeq (\sigma_{\mu} \circ \Phi, \rho^{\sigma_{\mu} \circ \Phi}, \mathcal{K}_{\mu})$$

which completes the proof.  $\square$

Let  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  be a polarizing system with a consistent representation  $(\sigma_{\mu}, \rho^{\sigma_{\mu}}, \mathcal{K}_{\mu})$ . From now on,  $(\pi, \rho^{\pi}, \mathcal{H})$  will denote the induced unitary representation

$$(6.22) \quad (\pi, \rho^{\pi}, \mathcal{H}) = \text{Ind}_{(M_0, \mathfrak{m})}^{(N_0, \mathfrak{n})} (\sigma_{\mu} \circ \Phi, \rho^{\sigma_{\mu} \circ \Phi}, \mathcal{K}_{\mu}).$$

Recall the definition of  $\mathfrak{a}[\mathfrak{n}]$  from Section 4.1. Since  $\mathfrak{m}$  is an ideal of  $\mathfrak{n}$ , we have

$$(6.23) \quad \mathfrak{m} \supseteq \mathfrak{a}[\mathfrak{n}]$$

because  $\mathfrak{m}$  contains all of the generators of  $\mathfrak{a}[\mathfrak{n}]$ .

**Lemma 6.14.**  $\mathfrak{a}[\mathfrak{n}]_0 \cap \mathcal{Z}(\mathfrak{n}) \subseteq \ker \lambda$ .

*Proof.* The proof of this lemma is essentially given throughout the proof of Case I of Theorem 6.4, and therefore here we only give a sketch of the proof. Let  $(\pi, \rho^{\pi}, \mathcal{H})$  be as in (6.22), and assume  $X \in \mathfrak{a}[\mathfrak{n}]_0 \cap \mathcal{Z}(\mathfrak{n})$  and  $\lambda(X) \neq 0$ . Using the definition of the induced representation, it is not difficult to see that  $\pi^{\infty}(X) \neq 0$ , which contradicts Proposition 4.2.  $\square$

**6.5. Relation between  $(\pi, \rho^{\pi}, \mathcal{H})$  and  $\lambda$ .** Let  $\lambda \in \mathfrak{n}_0^*$  such that  $B_{\lambda}$  is nonnegative definite. By Proposition 6.13 there exists a polarizing system  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$ , and by Theorem 6.4 the representation obtained by induction from a consistent representation of the polarizing system is irreducible. Our next task is to show that if we choose different polarizing systems, we always obtain the same representation.

**Proposition 6.15.** *Up to unitary equivalence and parity change, the representation  $(\pi, \rho^{\pi}, \mathcal{H})$  is uniquely determined by  $\lambda$ .*

*Proof.* We prove the proposition by induction on  $\dim \mathfrak{n}$ . The argument is similar to the proof of Theorem 6.4.

Let  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  (respectively,  $(M'_0, \mathfrak{m}', \Phi', C'_0, \mathfrak{c}', \lambda)$ ) be a polarizing system with a consistent representation  $(\sigma_{\mu}, \rho^{\sigma_{\mu}}, \mathcal{K}_{\mu})$  (respectively,  $(\sigma_{\mu'}, \rho^{\sigma_{\mu'}}, \mathcal{K}_{\mu'})$ ).

(Note that the two polarizing systems are associated to the same  $\lambda$ .) Suppose that  $(\pi, \rho^\pi, \mathcal{H})$  (respectively,  $(\pi', \rho^{\pi'}, \mathcal{H}')$ ) is the induced representation defined as in (6.22). Our main goal is to prove that

$$(6.24) \quad (\pi, \rho^\pi, \mathcal{H}) \simeq (\pi', \rho^{\pi'}, \mathcal{H}').$$

There are three cases to consider.

*Case I:  $(N_0, \mathfrak{n})$  is not reduced.* As in the proof of Case I in Theorem 6.4, we can show that

$$\mathfrak{a}[\mathfrak{n}] \cap \mathcal{Z}(\mathfrak{n}) \neq \{0\}.$$

Moreover,  $\mathfrak{a}[\mathfrak{n}] \cap \mathcal{Z}(\mathfrak{n}) \subseteq \mathfrak{m} \cap \mathfrak{m}'$ , and using Lemma 6.14 we can see that for every  $W \in \mathfrak{a}[\mathfrak{n}] \cap \mathcal{Z}(\mathfrak{n})$  we have  $\Phi(W) = 0$  and  $\Phi'(W) = 0$ .

Set  $\mathfrak{s} = \mathfrak{a}[\mathfrak{n}] \cap \mathcal{Z}(\mathfrak{n})$  and consider the corresponding sub super Lie group  $(S_0, \mathfrak{s})$  of  $(N_0, \mathfrak{n})$ . Since  $\mathfrak{s}$  is an ideal of  $\mathfrak{n}$ , we have a quotient homomorphism

$$\mathbf{q} : (N_0, \mathfrak{n}) \rightarrow (N_0/S_0, \mathfrak{n}/\mathfrak{s}).$$

The polarizing system  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  corresponds via  $\mathbf{q}$  to a polarizing system

$$(M_0/S_0, \mathfrak{m}/\mathfrak{s}, \Phi_{\mathbf{q}}, C_0, \mathfrak{c}, \lambda_{\mathbf{q}})$$

in  $(N_0/S_0, \mathfrak{n}/\mathfrak{s})$ , where  $\lambda_{\mathbf{q}} \in (\mathfrak{n}/\mathfrak{s})^*$  satisfies  $\lambda_{\mathbf{q}} \circ \mathbf{q} = \lambda$ . If we set

$$(\pi_{\mathbf{q}}, \rho^{\pi_{\mathbf{q}}}, \mathcal{H}_{\mathbf{q}}) = \text{Ind}_{(M_0/S_0, \mathfrak{m}/\mathfrak{s})}^{(N_0/S_0, \mathfrak{n}/\mathfrak{s})}(\sigma_{\mu} \circ \Phi_{\mathbf{q}}, \rho^{\sigma_{\mu} \circ \Phi_{\mathbf{q}}}, \mathcal{K}_{\mu})$$

then  $(\pi, \rho^\pi, \mathcal{H}) \simeq (\pi_{\mathbf{q}} \circ \mathbf{q}, \rho^{\pi_{\mathbf{q}} \circ \mathbf{q}}, \mathcal{H}_{\mathbf{q}})$ . From the other polarizing system and its consistent representation one can obtain another representation  $(\pi'_{\mathbf{q}}, \rho^{\pi'_{\mathbf{q}}}, \mathcal{H}'_{\mathbf{q}})$  of  $(N_0/S_0, \mathfrak{n}/\mathfrak{s})$  which is defined in a similar way. Since  $\dim \mathfrak{n}/\mathfrak{s} < \dim \mathfrak{n}$ , induction hypothesis implies that

$$(\pi_{\mathbf{q}}, \rho^{\pi_{\mathbf{q}}}, \mathcal{H}_{\mathbf{q}}) \simeq (\pi'_{\mathbf{q}}, \rho^{\pi'_{\mathbf{q}}}, \mathcal{H}'_{\mathbf{q}})$$

from which (6.24) follows immediately.

*Case II:  $(N_0, \mathfrak{n})$  is reduced and  $\mathcal{Z}(\mathfrak{n}) \cap \ker \lambda \neq \{0\}$ .* In this case  $\mathcal{Z}(\mathfrak{n}) \cap \ker \lambda$  is an ideal of  $\mathfrak{n}$  and  $\mathcal{Z}(\mathfrak{n}) \cap \ker \lambda \subseteq \mathfrak{m} \cap \mathfrak{m}'$ . Set  $\mathfrak{s} = \mathcal{Z}(\mathfrak{n}) \cap \ker \lambda$  and let  $(S_0, \mathfrak{s})$  be the corresponding sub super Lie group of  $(N_0, \mathfrak{n})$ . As in Case I above, using the quotient map

$$\mathbf{q} : (N_0, \mathfrak{n}) \rightarrow (N_0/S_0, \mathfrak{n}/\mathfrak{s})$$

we can obtain new polarizing systems and consistent representations for

$$(N_0/S_0, \mathfrak{n}/\mathfrak{s}).$$

The rest of the argument is similar to that of Case I above.

*Case III:  $(N_0, \mathfrak{n})$  is reduced and  $\mathcal{Z}(\mathfrak{n}) \cap \ker \lambda = \{0\}$ .* In this case the proof is very similar to that of Case III in Theorem 6.4. Without loss of generality we can assume that  $\mathfrak{n}$  is not of Clifford type. From  $\mathcal{Z}(\mathfrak{n}) \cap \ker \lambda = \{0\}$  it follows that  $\dim \mathcal{Z}(\mathfrak{n}) = 1$ . Let  $X, Y, Z$ , and  $\mathfrak{n}'$  be as in part (a) of Proposition 4.4. As shown in the proof of Case III in Theorem 6.4, we can choose  $X, Y, Z$  suitably such that there exist polarizing systems

$$(6.25) \quad (\overline{M}_0, \overline{\mathfrak{m}}, \overline{\Phi}, C_0, \mathfrak{c}, \overline{\lambda}) \quad \text{and} \quad (\overline{M}'_0, \overline{\mathfrak{m}}', \overline{\Phi}', C'_0, \mathfrak{c}', \overline{\lambda})$$

in  $(N_0, \mathfrak{n})$  with the following properties.

- (a)  $\bar{\lambda} = \text{Ad}^*(n)(\lambda)$  for some  $n \in N_0$ .
- (b)  $\bar{\mathfrak{m}} \subseteq \mathfrak{n}'$  and  $\bar{\mathfrak{m}}' \subseteq \mathfrak{n}'$ .
- (c)  $(\sigma_\mu, \rho^{\sigma_\mu}, \mathcal{K}_\mu)$  is consistent with  $(\bar{M}_0, \bar{\mathfrak{m}}, \bar{\Phi}, C_0, \mathfrak{c}, \bar{\lambda})$ .
- (d)  $(\sigma_{\mu'}, \rho^{\sigma_{\mu'}}, \mathcal{K}_{\mu'})$  is consistent with  $(\bar{M}'_0, \bar{\mathfrak{m}}', \bar{\Phi}', C'_0, \mathfrak{c}', \bar{\lambda})$ .
- (e) If  $(\bar{\pi}, \rho^{\bar{\pi}}, \bar{\mathcal{H}}) = \text{Ind}_{(\bar{M}_0, \bar{\mathfrak{m}})}^{(N_0, \mathfrak{n})}(\sigma_\mu \circ \bar{\Phi}, \rho^{\sigma_\mu \circ \bar{\Phi}}, \mathcal{K}_\mu)$  then

$$(\bar{\pi}, \rho^{\bar{\pi}}, \bar{\mathcal{H}}) \simeq (\pi, \rho^\pi, \mathcal{H}).$$

- (f) If  $(\bar{\pi}', \rho^{\bar{\pi}'}, \bar{\mathcal{H}}') = \text{Ind}_{(\bar{M}'_0, \bar{\mathfrak{m}}')}^{(N'_0, \mathfrak{n}')}(\sigma_{\mu'} \circ \bar{\Phi}', \rho^{\sigma_{\mu'} \circ \bar{\Phi}'}, \mathcal{K}_{\mu'})$  then

$$(\bar{\pi}', \rho^{\bar{\pi}'}, \bar{\mathcal{H}}') \simeq (\pi', \rho^{\pi'}, \mathcal{H}').$$

Let  $(N'_0, \mathfrak{n}')$  be the sub super Lie group of  $(N_0, \mathfrak{n})$  corresponding to  $\mathfrak{n}'$ . Since  $\dim \mathfrak{n}' < \dim \mathfrak{n}$ , by induction hypothesis we have

$$\text{Ind}_{(\bar{M}_0, \bar{\mathfrak{m}})}^{(N'_0, \mathfrak{n}')}(\sigma_\mu \circ \bar{\Phi}, \rho^{\sigma_\mu \circ \bar{\Phi}}, \mathcal{K}_\mu) \approx \text{Ind}_{(\bar{M}'_0, \bar{\mathfrak{m}}')}^{(N'_0, \mathfrak{n}')}(\sigma_{\mu'} \circ \bar{\Phi}', \rho^{\sigma_{\mu'} \circ \bar{\Phi}'}, \mathcal{K}_{\mu'})$$

and (6.24) follows by Proposition 3.1.  $\square$

**6.6. Geometric parametrization of representations.** Let  $(\pi, \rho^\pi, \mathcal{H})$  be an irreducible unitary representation of a nilpotent super Lie group  $(N_0, \mathfrak{n})$ . One can associate a coadjoint orbit  $\mathcal{O} \subseteq \mathfrak{n}_0^*$  to  $(\pi, \rho^\pi, \mathcal{H})$  as follows. Let  $(\pi, \mathcal{H})$  denote the restriction of  $(\pi, \rho^\pi, \mathcal{H})$  to  $N_0$ . From Corollary 6.3 it follows that  $(\pi, \mathcal{H})$  is a direct sum of finitely many copies of an irreducible unitary representation  $(\sigma, \mathcal{K})$  of  $N_0$ . By classical Kirillov theory [CG], the representation  $(\sigma, \mathcal{K})$  is associated to a coadjoint orbit  $\mathcal{O} \subseteq \mathfrak{n}_0^*$ . Theorem 6.2 shows that  $(\pi, \rho^\pi, \mathcal{H})$  is induced from a consistent representation of a polarizing system  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$  where  $\lambda \in \mathcal{O}$ .

Our last theorem puts together the results in this paper to obtain a geometric parametrization of irreducible unitary representations of  $(N_0, \mathfrak{n})$  by coadjoint orbits. Recall that

$$\mathfrak{n}_0^+ = \{ \lambda \in \mathfrak{n}_0^* \mid B_\lambda \text{ is nonnegative definite} \}.$$

**Theorem 6.16.** *For a nilpotent super Lie group  $(N_0, \mathfrak{n})$ , the process of associating a coadjoint orbit  $\mathcal{O} \subseteq \mathfrak{n}_0^*$  to an irreducible unitary representation  $(\pi, \rho^\pi, \mathcal{H})$  of  $(N_0, \mathfrak{n})$  yields a bijection between equivalence classes of irreducible unitary representations (up to unitary equivalence and parity change) and  $N_0$ -orbits in  $\mathfrak{n}_0^+$ .*

*Proof.* By part (a) of Theorem 6.2, any irreducible unitary representation  $(\pi, \rho^\pi, \mathcal{H})$  of  $(N_0, \mathfrak{n})$  is induced from a consistent representation of a polarizing system

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$$

and as shown in Section 6.4, it follows that  $\lambda \in \mathfrak{n}_0^+$ .

By part (b) of Theorem 6.2, if  $(\pi, \rho^\pi, \mathcal{H})$  is induced from a consistent representation of another polarizing system

$$(M'_0, \mathfrak{m}', \Phi', C'_0, \mathfrak{c}', \lambda')$$

then  $\lambda$  and  $\lambda'$  are in the same  $N_0$ -orbit. Moreover, once we fix a  $\lambda \in \mathfrak{n}_0^+$ , by Proposition 6.13 there always exists an associated polarizing system and a consistent representation, and by Proposition 6.15, up to unitary equivalence and parity change all such polarizing systems yield the same irreducible unitary representation of  $(N_0, \mathfrak{n})$ . □

**Remark.** One can actually prove that for every irreducible unitary representation  $(\pi, \rho^\pi, \mathcal{H})$  of  $(N_0, \mathfrak{n})$ , the space  $[\mathfrak{n}_1, [\mathfrak{n}_1, \mathfrak{n}_1]]$  acts trivially, i.e.,

$$(6.26) \quad \rho^\pi(X) = 0 \text{ for every } X \in [\mathfrak{n}_1, [\mathfrak{n}_1, \mathfrak{n}_1]].$$

Indeed if  $\lambda \in \mathfrak{n}_0^+$  then from Lemma 6.9, Lemma 6.10, and Lemma 6.11 it follows that  $[\mathfrak{n}_1, [\mathfrak{n}_1, \mathfrak{n}_1]] \subseteq \mathfrak{r}_\lambda$ . Consequently, when  $(\pi, \rho^\pi, \mathcal{H})$  is induced from a consistent representation of a polarizing system  $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$ , we have  $\Phi([\mathfrak{n}_1, [\mathfrak{n}_1, \mathfrak{n}_1]]) = 0$ . Statement (6.26) now follows from the realization of the induced representation given in Section 3.1 and the fact that  $[\mathfrak{n}_1, [\mathfrak{n}_1, \mathfrak{n}_1]]$  is  $N_0$ -invariant.

Another more direct way to prove (6.26) is to use the method of proof of Proposition 6.15. Statement (6.26) can be used to obtain slightly different proofs for the main results of this paper. We thank the referee for suggesting this statement and the second method of proof.

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